A variational method for second order shape derivatives

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Abstract

We consider shape functionals obtained as minima on Sobolev spaces of classical integrals having smooth and convex densities, under mixed Dirichlet-Neumann boundary conditions. We propose a new approach for the computation of the second order shape derivative of such functionals, yielding a general existence and representation theorem. In particular, we consider the p-torsional rigidity functional for $p \geq 2$.

1 Introduction

Aim of this paper is to provide a new method for the computation of second order shape derivatives, which applies to a broad class of shape functionals associated with classical problems in the Calculus of Variations. Let us recall that, if $J(\cdot)$ is a functional depending on a subset Ω of $\mathbb{R}^n$ and V is a deformation field in $C^1(\mathbb{R}^n; \mathbb{R}^n)$, the first and second order shape derivatives of J at Ω in direction V, if they exist, are given respectively by the limits

$$J'(\Omega, V) := \lim_{\varepsilon \to 0} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon}, \qquad J''(\Omega, V) := \lim_{\varepsilon \to 0} 2 \frac{J(\Omega_\varepsilon) - J(\Omega) - \varepsilon J'(\Omega, V)}{\varepsilon^2},$$

where Ω_ε are the perturbed domains $\Omega_\varepsilon := \Psi_\varepsilon(\Omega)$, with $\Psi_\varepsilon(x) := x + \varepsilon V(x)$.

The literature on shape derivatives is very wide, and in recent years it has seen a rapid flourishing, also stimulated by the advances in numerical methods for the determination of optimal shapes. We limit ourselves to quote the monographs [17, 25, 42], where a lot of bibliographical references in this field can be found. Concerning in particular second order shape derivatives, their computation is usually irksome, but often deserves some efforts because the study of their sign allows to detect whether a critical shape, namely a domain with vanishing first order shape derivative, is actually an optimal one. As contributions in this direction, let us mention without any attempt of completeness the papers [1, 10, 16, 20, 23, 24, 29, 36]. In particular, in [36] Novruzi and Pierre proved a quite general and helpful structure result about second order shape derivatives. However, this requires the a priori assumption that they exist, and demands that the boundary of the domain Ω where

the derivatives are computed is sufficiently smooth (for the structure of shape derivatives around irregular domains, see [29]). Moreover, the linear and bilinear forms which appear in this structure result have to be identified each time for the particular functional under study, and this turns out to be a delicate task (see the discussion in [25, Section 5.9.4]).

In this work we propose a new approach to second order shape derivatives, which applies under rather mild regularity assumptions to shape functionals of the form

$$J(\Omega) := -\inf \left\{ \int_{\Omega} [f(\nabla u) + g(u)]\, dx \ : \ u \in H(\Omega) \right\}. \tag{1.1}$$

Here f and g are assumed to be convex and smooth functions satisfying suitable growth conditions, while $H(\Omega)$ indicates the space of functions in $H^1(\Omega)$ which satisfy the Dirichlet condition $u = 0$ on a fixed, nonempty, measurable portion $\Gamma_D \subset \partial\Omega$; the Neumann part of the boundary, *i.e.* the complement $\partial\Omega \setminus \Gamma_D$, will be denoted by Γ_N.

As a natural continuation of our previous paper [8], where we introduced a new approach to first order shape derivatives for functionals of the type (1.1), here we tackle second order shape derivatives by the same method. The main results obtained in [8] are briefly recalled in Section 2, see eq. (2.8)-(2.9). The basic ingredients we employ are the analysis of the Γ-convergence of the differential quotients appearing in the definition of $J''(\Omega, V)$, and the duality principle

$$J(\Omega) = J^*(\Omega) := \inf \left\{ \int_{\Omega} [f^*(\sigma) + g^*(\operatorname{div}\sigma)]\, dx \ : \ \sigma \in X(\Omega; \mathbb{R}^n) \right\}, \tag{1.2}$$

where $X(\Omega; \mathbb{R}^n)$ is a suitable space of vector fields.

Our main results, which are intimately related to each other, are:

– an existence result for $J''(\Omega, V)$ which is a quadratic form in V and can be represented as follows:

$$J''(\Omega, V) = \int_{\partial\Omega} \left(\mathcal{C}(\overline{u}, V) \cdot n \right) d\mathcal{H}^{n-1} + q(\overline{u}, V); \tag{1.3}$$

here $\mathcal{C}(\overline{u}, V) \cdot n$ is the normal trace of a suitable vector field (see (3.4)) depending on the solution $\overline{u}$ to $J(\Omega)$ and quadratically on the deformation V, whereas $q(\overline{u}, V)$ is a nonlocal term which involves a further vector field $B(\overline{u}, V)$ and a quadratic form $\mathcal{Q}(\overline{u}, \cdot)$ depending on the second derivatives of f and g (see Theorem 3.4); we stress that, as detailed in Remark 3.6 below, formula (1.3) fits the general representation result for second order shape derivatives given in [36, Corollary 2.4];

– a regularity result of type $W^{2,2}_{\mathrm{loc}}$ for the solution to $J(\Omega)$ (see Proposition 3.2);

– a new necessary optimality condition, which involves the distributional divergence of the above mentioned vector field $B(\overline{u}, V)$ (see Proposition 3.3).

Both the regularity of the solution and the optimality condition on one hand seem to have an independent interest, and on the other hand are strictly related to the second order shape derivative. Namely, if the $W^{2,2}_{\mathrm{loc}}$ regularity of $\overline{u}$ stated in Proposition 3.2 extends up to the boundary, the field $\mathcal{C}(\overline{u}, V)$ turns out to admit a normal trace on the boundary as soon as the latter is piecewise C^1; moreover, in order to arrive at the expression (1.3) for $J''(\Omega, V)$, the optimality condition given in Proposition 3.3 is exploited as a crucial tool.

The paper is organized as follows. After providing some notation and preliminary background in Section 2, we state in Section 3 our main results (Proposition 3.2, Proposition 3.3 and Theorem 3.4),

followed by some comments and examples. In fact, we show that we can recover straightforward the second order shape derivative of the torsional rigidity, as given *e.g.* in [25, Section 5.9.6], and we are able to extend the formula to the case of mixed Dirichlet-Neumann conditions. Moreover, we consider the case of p-torsional rigidity for $p > 2$. In this case, to the best of our knowledge, only the first order shape derivative is available in the literature (see [14, 21]). We are able to compute the second order shape derivative under an additional assumption, which is quite technical and is related to a deep regularity issue for the p-Laplace operator; checking its validity seems to be a delicate problem of independent interest which goes beyond the scope of this paper.

The next three sections are devoted to the proofs: the existence of the second order shape derivative (Theorem 3.4 (i)) is achieved in Section 4; the regularity result of Proposition 3.2 and the optimality conditions of Proposition 3.3 are proved in Section 5; eventually, in Section 6 we prove the representation formula for the second order shape derivative (Theorem 3.4 (ii)).

In Section 7 we present some variants of our results, including in particular the case of the p-torsional rigidity for $p \in [2, +\infty)$, and then we conclude the paper by addressing some possible perspectives. Section 8 is an appendix where some auxiliary lemmas and technical facts are collected.

2 Preliminaries

Standing assumptions. Unless otherwise stated, we work under the following hypotheses, which will be referred to as standing assumptions:

- Ω is an open bounded connected set, with a piecewise C^1 boundary and unit outer normal n;

- $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are of class C^2 and are strongly convex, namely there exist positive constants m and k such that

$$(\nabla^2 f - mI) \text{ is positively semidefinite}, \qquad g'' - k \geq 0; \qquad (2.1)$$

- f and g satisfy the growth conditions

$$\begin{cases} C_1(|z|^2 - 1) \leq f(z) \leq C_2(|z|^2 + 1) & \forall z \in \mathbb{R}^n \\ C_3(|v|^2 - 1) \leq g(v) \leq C_4(|v|^q + 1) & \forall v \in \mathbb{R}, \end{cases} \qquad (2.2)$$

  where C_i are positive constants, while the exponent q is assumed to satisfy $q = 2^* := \frac{2n}{n-2}$ if $n > 2$ and $q \in (1, +\infty)$ if $n \leq 2$;

- $g(0) = 0$ (which is not restrictive up to a translation);

- the solution $\overline{u}$ to $J(\Omega)$ is Lipschitz (we point out that, by strict convexity, $\overline{u}$ is unique, and we refer to [11, 22, 31, 32] for some Lipschitz regularity results);

- the deformation field V belongs to $C^1(\mathbb{R}^n; \mathbb{R}^n)$.

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Some standard notation. We adopt the convention of repeated indices. Given two vectors a, b in $\mathbb{R}^n$ we use the notation $\langle a, b \rangle$ to denote their Euclidean scalar product; moreover we denote by $a \otimes b$ the tensor product of a and b, namely the matrix $(a \otimes b)_{ij} := a_i b_j$. Given two matrices A and B in $\mathbb{R}^{n \times n}$, we write $A : B$ to denote their Euclidean scalar product, namely $A : B = A_{ij} B_{ij}$. We denote by A^{-1} and A^T the inverse and the transpose matrices of A, by A^{-T} the transposition of the inverse of A and by I the identity matrix. Moreover, we denote by $a_k(A)$ the k-th invariant of the matrix A, in particular

$$a_1(A) = \operatorname{tr}(A) \,, \qquad a_2(A) = \frac{1}{2} \big(\operatorname{tr}(A) I - A^T \big) : A \,, \qquad a_n(A) = \det A \,.$$

For a tensor field $A \in C^1(\mathbb{R}^n; \mathbb{R}^{n \times n})$, by $\operatorname{div} A$ we mean its divergence with respect to lines, namely $(\operatorname{div} A)_i := \partial_j A_{ij}$.

Let Ω be as in the standing assumptions. Given a vector field V on $\partial\Omega$, we decompose it into a *tangential and normal component* as $V = V_\Gamma + V_n n$, with $V_n := \langle V, n \rangle$.

We recall that, for $\varphi \in C^1(\partial\Omega)$ and $V \in C^1(\partial\Omega; \mathbb{R}^n)$, the *tangential gradient* of φ and the *tangential divergence* of V are given respectively by $\nabla_\Gamma \varphi := \nabla\tilde\varphi - \langle \nabla\tilde\varphi, n \rangle n$ and $\operatorname{div}_\Gamma V := \operatorname{div} \widetilde{V} - \langle D\widetilde{V} n, n \rangle$ on $\partial\Omega$, where $\tilde\varphi$ and $\widetilde{V}$ are arbitrary C^1 extensions to $\mathbb{R}^n$ of φ and V. In particular, if Ω is of class C^2, the tangential divergence of n gives the scalar mean curvature, that is $H_{\partial\Omega} := \operatorname{div}_\Gamma n$. The first and second order normal derivatives of φ will be denoted by $\partial_n \varphi := \langle \nabla\varphi, n \rangle$ and $\partial^2_{nn} \varphi := \langle (\nabla^2 \varphi) n, n \rangle$. Given a vector field $\Psi \in L^2(\Omega; \mathbb{R}^n)$ with distributional divergence in $L^2(\Omega)$, we denote by $\Psi \cdot n$ its *normal trace* on $\partial\Omega$, meant as the unique element in $H^{-1/2}(\partial\Omega)$ such that

$$\int_{\partial\Omega} (\Psi \cdot n) \, \varphi \, d\mathcal{H}^{n-1} = \int_\Omega \big(\langle \Psi, \nabla\varphi \rangle + \varphi \operatorname{div} \Psi \big) \, dx \qquad \forall \varphi \in H^1(\Omega) \,, \tag{2.3}$$

where the boundary integral at the l.h.s. is intended as the duality bracket between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. For a detailed account on the theory of weak traces, we refer to [3, 12] (see also [8, Section 2.1]).

Existence of solutions, dual formulation and optimality conditions. Under the standing assumptions, the infimum problem $J(\Omega)$ admits a unique solution (*cf.* [8, Lemma 2.1]), denoted by $\overline{u}$, in the space

$$H(\Omega) := \{ u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma_D \} \,.$$

Moreover, by standard duality arguments (*cf.* [8, Lemma 2.2]), it holds $J(\Omega) = J^*(\Omega)$, where $J^*(\Omega)$ is the infimum problem introduced in (1.2), set over the space

$$X(\Omega; \mathbb{R}^n) := \{ \sigma \in L^2(\Omega; \mathbb{R}^n) \ : \ \operatorname{div} \sigma \in L^{q'}(\Omega) \,, \ \sigma \cdot n = 0 \text{ on } \Gamma_N \} \,, \tag{2.4}$$

being $q' := q/(q-1)$. Also $J^*(\Omega)$ admits a unique solution, denoted by $\overline{\sigma}$, which is related to $\overline{u}$ by the following differential equalities:

$$\begin{cases} \overline{\sigma} = \nabla f(\nabla\overline{u}) \quad \text{a.e. in } \Omega \\ \\ \operatorname{div} \overline{\sigma} = g'(\overline{u}) \quad \text{a.e. in } \Omega \end{cases} \quad \text{or equivalently} \quad \begin{cases} \nabla\overline{u} = \nabla f^*(\overline{\sigma}) \quad \text{a.e. in } \Omega \\ \\ \overline{u} = (g^*)'(\operatorname{div}\overline{\sigma}) \quad \text{a.e. in } \Omega \,. \end{cases} \tag{2.5}$$

Note that, by (2.5) and in view of the regularity assumed on f, g and $\overline{u}$, we have $\nabla\overline{u}, \overline{\sigma} \in L^\infty(\Omega; \mathbb{R}^n)$ and $\overline{u}, \operatorname{div}\overline{\sigma} \in L^\infty(\Omega)$.

First order shape derivative. Let $V \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ be a deformation field and set for every ε

$$\Omega_\varepsilon := \Psi_\varepsilon(\Omega), \quad \text{being } \Psi_\varepsilon(x) := x + \varepsilon V(x). \tag{2.6}$$

Under the standing assumptions, the first order shape derivative of J at Ω in direction V, defined by

$$J'(\Omega, V) := \lim_{\varepsilon \to 0} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon}, \tag{2.7}$$

exists and is given by

$$J'(\Omega, V) = \int_\Omega A(\overline{u}) : DV \, dx = \int_{\partial\Omega} \langle A(\overline{u}) \, n, V \rangle \, d\mathcal{H}^{n-1}, \tag{2.8}$$

where $A(\overline{u})$ denotes the following tensor which turns out to be divergence free on Ω with a normal trace $A(\overline{u}) \, n$ in $L^\infty(\partial\Omega)$:

$$A(\overline{u}) = \nabla\overline{u} \otimes \overline{\sigma} - (f(\nabla\overline{u}) + g(\overline{u})) \, I = \nabla\overline{u} \otimes \overline{\sigma} + (f^*(\overline{\sigma}) + g^*(\operatorname{div}\overline{\sigma}) - \langle\nabla\overline{u}, \overline{\sigma}\rangle - \overline{u}\operatorname{div}\overline{\sigma}) \, I \quad (2.9)$$

(with $\overline{\sigma} = \nabla f(\nabla\overline{u})$). For the the proof of (2.8), see Theorem 3.3 and Theorem 3.7 in [8]; actually, even though such results are not stated for the case of mixed Dirichlet-Neumann boundary conditions, it can be easily checked that their proofs continue to work unaltered in such case. Moreover, if $\overline{u}$ is Lipschitz and $\nabla\overline{u}, \overline{\sigma}$ are in $BV(\Omega; \mathbb{R}^n)$, the boundary integral in (2.8) can be rewritten as a linear expression of V_n, namely it holds $J'(\Omega, V) = l_1(V_n)$ with

$$l_1(\varphi) := \int_{\Gamma_D} f^*(\overline{\sigma}) \, \varphi \, d\mathcal{H}^{n-1} - \int_{\Gamma_N} (f(\nabla\overline{u}) + g(\overline{u})) \, \varphi \, d\mathcal{H}^{n-1}.$$

Reformulation from a variable domain to a fixed domain. Following the same procedure adopted in [8, Lemma 4.1] in order to recast the first order shape derivative $J'(\Omega, V)$, we rewrite the variational problem $J(\Omega_\varepsilon)$ over the fixed domain Ω: by using a standard change of variables, we obtain

$$J(\Omega_\varepsilon) = -\inf_{u \in H(\Omega)} \int_\Omega [f_\varepsilon(\nabla u) + g_\varepsilon(u)] \, dx \tag{2.10}$$

where f_ε and g_ε are given by

$$f_\varepsilon(x, z) := f(D\Psi_\varepsilon^{-T} z)\beta_\varepsilon, \qquad g_\varepsilon(x, v) := g(v)\beta_\varepsilon \tag{2.11}$$

being β_ε the Jacobian of the map Ψ_ε. Note that, in view of (2.6), β_ε can be written as a polynomial in ε, with the invariants of DV as coefficients:

$$\beta_\varepsilon := |\det D\Psi_\varepsilon| = 1 + a_1(DV)\varepsilon + a_2(DV)\varepsilon^2 + \cdots + a_n(DV)\varepsilon^n. \tag{2.12}$$

In a similar way, the dual variational problems $J^*(\Omega_\varepsilon)$ can be rewritten as

$$J^*(\Omega_\varepsilon) = \inf_{\sigma \in X(\Omega; \mathbb{R}^n)} \int_\Omega [f_\varepsilon^*(\sigma) + g_\varepsilon^*(\operatorname{div}\sigma)] \, dx, \tag{2.13}$$

where f_ε^* and g_ε^* are the Fenchel conjugates of f_ε and g_ε with respect to the second variable, which by direct computation satisfy

$$f_\varepsilon^*(x, z) = f^*(\beta_\varepsilon^{-1} D\Psi_\varepsilon z)\beta_\varepsilon, \qquad g_\varepsilon^*(x, \xi) = g^*(\beta_\varepsilon^{-1}\operatorname{div}\xi)\beta_\varepsilon. \tag{2.14}$$

In the subsequent asymptotic analysis as $\varepsilon \to 0$, we shall exploit the following developments holding for small ε, being Ψ_ε and β_ε defined respectively by (2.6) and (2.12):

$$D\Psi_\varepsilon^{-T} = I - \varepsilon DV^T + \varepsilon^2 (DV^T)^2 + \varepsilon^3 M_\varepsilon\,, \quad \text{with } \sup_\varepsilon \|M_\varepsilon\| \leq C\,, \tag{2.15}$$

$$\beta_\varepsilon = 1 + \varepsilon \operatorname{div} V + \varepsilon^2 a_2(DV) + \varepsilon^3 m_\varepsilon\,, \quad \text{with } \sup_\varepsilon \|m_\varepsilon\| \leq C\,, \tag{2.16}$$

$$\beta_\varepsilon^{-1} = 1 - \varepsilon \operatorname{div} V + \varepsilon^2 ((\operatorname{div} V)^2 - a_2(DV)) + \varepsilon^3 \widetilde{m_\varepsilon}(V)\,, \quad \text{with } \sup_\varepsilon \|\widetilde{m_\varepsilon}\| \leq C\,. \tag{2.17}$$

3 Main results

All the results in this Section concern the shape functional J defined in (1.1) and are stated under the standing assumptions given in Section 2. Our goal is to provide an existence and representation result for the second order shape derivative of J, meant according to the following

Definition 3.1. *Given $V \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, J is said to be second order differentiable at Ω in the direction V, if the following limit exists:*

$$J''(\Omega, V) := \lim_{\varepsilon \to 0} 2\, \frac{J(\Omega_\varepsilon) - J(\Omega) - \varepsilon J'(\Omega, V)}{\varepsilon^2}\,, \tag{3.1}$$

where J' is the first order shape derivative of J at Ω in direction V (see (2.7) and (2.8) for its definition and representation formula).

To prepare our main result, some preliminaries are in order. The next two propositions will be necessary in order to prove the main theorem, but also seem to have an autonomous interest: as mentioned in the Introduction, the former concerns the regularity of the unique solution $\overline{u}$ to $J(\Omega)$ (for $W^{2,2}$-type regularity results for solutions to quasilinear elliptic equations, see for instance [15, 41, 40] and references therein); the latter states a new necessary condition for optimality.

Proposition 3.2. (regularity of the solution)
Under the standing assumptions, $\overline{u}$ belongs to $W^{2,2}_{\mathrm{loc}}(\Omega)$.

Proposition 3.3. (necessary optimality condition)
Under the standing assumptions, the vector field defined by

$$B(\overline{u}, V) := \nabla^2 f(\nabla \overline{u})(\nabla^2 \overline{u})V - (DV - \operatorname{div} V\, I)\overline{\sigma} \tag{3.2}$$

satisfies

$$\operatorname{div}\left[B(\overline{u}, V)\right] = g''(\overline{u})\langle V, \nabla \overline{u}\rangle + g'(\overline{u}) \operatorname{div} V \qquad \text{in } \mathcal{D}'(\Omega)\,. \tag{3.3}$$

The vector field $B(\overline{u}, V)$ defined in (3.2) will appear in the representation formula for the second order shape derivative. Besides $B(\overline{u}, V)$, such formula will involve another vector field depending on $\overline{u}$ and V, and a quadratic form on the space $H(\Omega)$, which are defined respectively as follows. We let $\mathcal{C}(\overline{u}, V)$ be the following vector field, which depends quadratically on V:

$$\begin{aligned}
\mathcal{C}(\overline{u}, V) := \ & -\langle V, \nabla \overline{u}\rangle (\nabla^2 f(\nabla \overline{u}))(\nabla^2 \overline{u})\, V - \langle V, \nabla \overline{u}\rangle (\operatorname{div} V\, I - DV)\nabla f(\nabla \overline{u}) + \\[4pt]
& + \langle DV\, \nabla f(\nabla \overline{u}), \nabla \overline{u}\rangle V - \langle DV\, V, \nabla \overline{u}\rangle \nabla f(\nabla \overline{u}) + \\[4pt]
& - \big(f(\nabla \overline{u}) + g(\overline{u})\big)(\operatorname{div} V\, I - DV)V\,.
\end{aligned} \tag{3.4}$$

We emphasize that the distributional Hessian of $\overline{u}$ appearing in (3.2) and (3.4) belongs to $L^2_{\mathrm{loc}}(\Omega)$ thanks to Proposition 3.2.

We let $\mathcal{Q}(\overline{u}, \cdot)$ be the quadratic form given by

$$\mathcal{Q}(\overline{u}, w) := \int_\Omega 2\left[Q_f(x, \nabla w) + Q_g(x, w)\right] dx \qquad \forall w \in H(\Omega), \tag{3.5}$$

where $Q_f(x, \cdot)$ and $Q_g(x, \cdot)$ denote the quadratic integrands

$$Q_f(x, z) := \frac{1}{2} \langle \nabla^2 f(\nabla \overline{u}(x)) z, z \rangle \quad \forall z \in \mathbb{R}^n, \qquad Q_g(x, v) := \frac{1}{2} g''(\overline{u}(x)) v^2 \quad \forall v \in \mathbb{R}.$$

In the sequel for brevity we shall omit to denote the dependence of Q_f and of Q_g on x and we shall simply write $Q_f(\cdot)$ and $Q_g(\cdot)$ in place of $Q_f(x, \cdot)$ and $Q_g(x, \cdot)$.

We are now in a position to state our main result.

Theorem 3.4. (existence and representation of the second order shape derivative)

Under the standing assumptions there hold:

(i) the second order shape derivative $J''(\Omega, V)$ defined in (3.1) exists and it is a quadratic form in V;

(ii) if $\overline{u}$ is $W^{2,2}(\Omega)$, the vector fields $\mathcal{C}(\overline{u}, V)$ and $B(\overline{u}, V)$, defined respectively in (3.4) and (3.2), admit a normal trace in $H^{-1/2}(\partial\Omega)$, and $J''(\Omega, V)$ is given by

$$J''(\Omega, V) = \int_{\partial\Omega} \left(\mathcal{C}(\overline{u}, V) \cdot n\right) d\mathcal{H}^{n-1} + q(\overline{u}, V), \tag{3.6}$$

where $q(\overline{u}, V)$ is the following non local term:

$$q(\overline{u}, V) := -\inf_{w \in H(\Omega)} \left\{ \mathcal{Q}\big(\overline{u}, w - \langle V, \nabla \overline{u} \rangle\big) + 2 \int_{\partial\Omega} \left(w - \langle V, \nabla \overline{u} \rangle\right) B(\overline{u}, V) \cdot n \, d\mathcal{H}^{n-1} \right\}. \tag{3.7}$$

Remark 3.5. Let us show how the representation formula for $J''(\Omega, V)$ given by (3.6) can be rewritten if one prefers to separate the contributions coming from the Dirichlet and Neumann portions of the boundary. First observe that, since $\overline{u}$ is assumed to be $W^{2,2}$ up to the boundary, the product $\langle V, \nabla \overline{u} \rangle$ belongs to $H^1(\Omega)$; thus, by considering the traslation $v := w - \langle V, \nabla \overline{u} \rangle$ in (3.7), the nonlocal term $q(\overline{u}, V)$ can be recast as

$$q(\overline{u}, V) = -\inf_{\substack{v \in H^1(\Omega) \\ v = -V_n \partial_n \overline{u} \text{ on } \Gamma_D}} \left\{ \mathcal{Q}(\overline{u}, v) + 2 \int_{\Gamma_N} v \, B(\overline{u}, V) \cdot n \, d\mathcal{H}^{n-1} \right\} + 2 \int_{\Gamma_D} \langle V, \nabla \overline{u} \rangle \, B(\overline{u}, V) \cdot n \, d\mathcal{H}^{n-1}. \tag{3.8}$$

By combining (3.6) and (3.8), we obtain

$$\begin{aligned} J''(\Omega, V) \;=\; & \int_{\Gamma_D} \big(\mathcal{C}(\overline{u}, V) + 2\langle V, \nabla \overline{u} \rangle \, B(\overline{u}, V)\big) \cdot n \, d\mathcal{H}^{n-1} + \int_{\Gamma_N} \big(\mathcal{C}(\overline{u}, V) \cdot n\big) \, d\mathcal{H}^{n-1} + \\ & - \inf_{\substack{v \in H^1(\Omega) \\ v = -V_n \partial_n \overline{u} \text{ on } \Gamma_D}} \left\{ \mathcal{Q}(\overline{u}, v) + 2 \int_{\Gamma_N} v \, B(\overline{u}, V) \cdot n \, d\mathcal{H}^{n-1} \right\}; \end{aligned} \tag{3.9}$$

in Remark 3.6 below we will see that, under suitable regularity assumptions, the infimum problem appearing in (3.9) is a quadratic form in V_n.

Next, in order to further simplify (3.9), we observe that the boundary integrals appearing therein depend only on the normal traces of $\mathcal{C}(\overline{u}, V)$ and $B(\overline{u}, V)$ on $\partial\Omega$. Therefore, such fields can be replaced by any simpler vector field with the same normal trace, which can be chosen in different ways on the Dirichlet and Neumann portions of the boundary. On Γ_D, since $\nabla\overline{u}$ is parallel to n on $\partial\Omega$, $\mathcal{C}(\overline{u}, V) + 2\langle V, \nabla\overline{u}\rangle B(\overline{u}, V)$ has the same normal trace as

$$\mathcal{C}_D(\overline{u}, V) := \langle V, \nabla\overline{u}\rangle(\nabla^2 f(\nabla\overline{u}))(\nabla^2\overline{u}) V + (\langle\nabla f(\nabla\overline{u}), \nabla\overline{u}\rangle - f(\nabla\overline{u}))(\operatorname{div} V I - DV)V; \quad (3.10)$$

while on Γ_N, since $\overline{\sigma} \cdot n = 0$ on such portion of the boundary, $\mathcal{C}(\overline{u}, V)$ has the same normal trace as

$$\mathcal{C}_N(\overline{u}, V) := -\langle V, \nabla\overline{u}\rangle(\nabla^2 f(\nabla\overline{u}))(\nabla^2\overline{u}) V + \langle V, \nabla\overline{u}\rangle DV\nabla f(\nabla\overline{u}) +$$

$$+\langle DV\,\nabla f(\nabla\overline{u}), \nabla\overline{u}\rangle V - (f(\nabla\overline{u}) + g(\overline{u}))(\operatorname{div} V I - DV)V.$$

We end up with the following reformulation of (3.6):

$$
\begin{aligned}
J''(\Omega, V) &= \int_{\Gamma_D} (\mathcal{C}_D(\overline{u}, V) \cdot n)\, d\mathcal{H}^{n-1} + \int_{\Gamma_N} (\mathcal{C}_N(\overline{u}, V) \cdot n)\, d\mathcal{H}^{n-1} + \\
&\quad - \inf_{\substack{v \in H^1(\Omega) \\ v = -V_n\partial_n\overline{u} \text{ on } \Gamma_D}} \left\{ \mathcal{Q}(\overline{u}, v) + 2\int_{\Gamma_N} v\, B(\overline{u}, V) \cdot n\, d\mathcal{H}^{n-1} \right\};
\end{aligned}
\quad (3.11)
$$

note that we readily obtain the expression of the second order shape derivative in the pure Dirichlet case, i.e. when $\Gamma_D = \partial\Omega$, by dropping the integrals over Γ_N in the above equality.

Remark 3.6. It is interesting to compare our representation formula (3.6) with the structure result proved in [36], which allows to enlighten the role played by the tangential and the normal components of the deformation field V on $\partial\Omega$. Notice that, in general, these components are not decoupled, i.e. $J''(\Omega, V) \neq J''(\Omega, V_\Gamma) + J''(\Omega, V_n n)$. In fact, according to the structure theorem given in [36, Corollary 2.4, Remark 2.10] (see also [25, Theorem 5.9.2]), under suitable regularity conditions $J''(\Omega, V)$ can be decomposed as $J''(\Omega, V) = l_1(z) + l_2(V_n)$, where l_2 is a quadratic form on $C^1(\partial\Omega)$ and l_1 is the linear form on $C^0(\partial\Omega)$ associated with the first order shape derivative, evaluated in

$$z := \langle V_\Gamma, Dn\, V_\Gamma\rangle - 2\langle V_\Gamma, \nabla_\Gamma V_n\rangle; \quad (3.12)$$

in this way the scalar z encodes the coupling between the tangential and normal components of V. In the case of our functional J, by Theorem 3.7 in [8], the linear form l_1 is given by

$$l_1(\varphi) = \int_{\Gamma_D} f^*(\overline{\sigma})\,\varphi\, d\mathcal{H}^{n-1} - \int_{\Gamma_N} (f(\nabla\overline{u}) + g(\overline{u}))\,\varphi\, d\mathcal{H}^{n-1}, \quad (3.13)$$

and we claim that, as soon as $\partial\Omega$ is C^2, by Theorem 3.4 the quadratic form l_2 is given by

$$l_2(\varphi) \quad := \int_{\Gamma_D} \varphi^2 \left[\partial_n\overline{u}\langle D\overline{\sigma}n, n\rangle + f^*(\overline{\sigma})H_{\partial\Omega} \right] d\mathcal{H}^{n-1} +$$

$$+ \int_{\Gamma_N} \left[\langle \overline{\sigma}, \nabla_\Gamma(\varphi^2 \partial_n\overline{u}) \rangle - \varphi^2 \left[\langle (\nabla^2\overline{u})\,\overline{\sigma}, n\rangle + (f(\nabla\overline{u}) + g(\overline{u}))H_{\partial\Omega} + \partial_n\overline{u}\langle D\overline{\sigma}n, n\rangle \right] \right] d\mathcal{H}^{n-1} +$$

$$- \inf_{\substack{v\in H^1(\Omega) \\ v=-\varphi\partial_n\overline{u} \text{ on } \Gamma_D}} \left\{ \mathcal{Q}(\overline{u}, v) + 2 \int_{\Gamma_N} v \left[\varphi\langle D\overline{\sigma}n, n\rangle - \langle \overline{\sigma}, \nabla_\Gamma\varphi\rangle \right] d\mathcal{H}^{n-1} \right\}.$$

(3.14)

This claim can be easily checked when $V_\Gamma = 0$ and $\partial\Omega = \Gamma_D$. Indeed in this case we have $z = 0$ and hence $l_1(z) = 0$; moreover, starting from equality (3.6), if we replace $\mathcal{C}(\overline{u}, V)$ by the simpler field $\mathcal{C}_D(\overline{u}, V)$ given in (3.10) and we use the identity $\langle(\operatorname{div} VI - DV)n, n\rangle = V_n H_{\partial\Omega}$ (holding thanks to the C^2 regularity assumption on $\partial\Omega$), we are led to the equality $J''(\Omega, V) = l_2(V_n)$, with l_2 defined according to (3.14). The proof of the claim in the general case requires some tedious but straightforward computations, that we omit for the sake of conciseness.

Remark 3.7. We point out that the nonlocal part of the second variation, namely the term $q(\overline{u}, V)$ defined in (3.7), may also be reformulated in dual form. In analogy with (3.5), let us introduce the quadratic form $\mathcal{Q}^*(\overline{\sigma}, \cdot)$ on the space $X(\Omega; \mathbb{R}^n)$ defined in (2.4): we set

$$\mathcal{Q}^*(\overline{\sigma}, \eta) := \int_\Omega 2[Q_{f^*}(x, \eta) + Q_{g^*}(x, \operatorname{div}\eta)]\, dx\,,$$

with $Q_{f^*}(x, z) := \frac{1}{2}\langle \nabla^2 f^*(\overline{\sigma}(x))z, z\rangle$ and $Q_{g^*}(x, v) := \frac{1}{2}(g^*)''(\operatorname{div}\overline{\sigma}(x))v^2$. (For brevity in the sequel we adopt the notation $Q_{f^*}(\cdot)$ and $Q_{g^*}(\cdot)$ in place of $Q_{f^*}(x, \cdot)$ and $Q_{g^*}(x, \cdot)$.)
A standard duality argument (see Lemma 8.5) yields

$$q(\overline{u}, V) = \inf_{\eta\in X(\Omega; \mathbb{R}^n)} \left\{ \mathcal{Q}^*(\overline{\sigma}, \eta - B(\overline{u}, V)) + \int_{\Gamma_D} 2\, \partial_n\overline{u}\, V_n\,(\eta\cdot n)\, d\mathcal{H}^{n-1} \right\},$$

or equivalently, via a translation,

$$q(\overline{u}, V) = \inf_{\eta\in X(\Omega; \mathbb{R}^n)-B(\overline{u}, V)} \left\{ \mathcal{Q}^*(\overline{\sigma}, \eta) + \int_{\Gamma_D} 2\, \partial_n\overline{u}\, V_n\,(\eta\cdot n)\, d\mathcal{H}^{n-1} \right\} + \int_{\Gamma_D} 2\, \partial_n\overline{u}\, V_n\,(B(\overline{u}, V)\cdot n)\, d\mathcal{H}^{n-1}\,.$$

Example 3.8. An example of shape functional which is covered by Theorem 3.4 is the torsional rigidity under mixed Dirichlet-Neumann boundary conditions:

$$J(\Omega) = -\inf\left\{ \int_\Omega \left(\frac{|\nabla u|^2}{2} - \lambda u \right) dx\ :\ u\in H(\Omega) \right\}.$$

In this case, assuming $\partial\Omega$ of class C^2 and taking deformations normal to the boundary, the second order shape derivative given in (3.11) reads

$$J''(\Omega, V) = \quad -\frac{1}{2}\int_{\Gamma_D} V_n^2 \left(2\lambda\partial_n\overline{u} + |\partial_n\overline{u}|^2 H_{\partial\Omega} \right) d\mathcal{H}^{n-1} +$$

$$-\frac{1}{2}\int_{\Gamma_N} V_n^2 \left[(-2\lambda\overline{u} + |\nabla\overline{u}|^2)H_{\partial\Omega} + 2\langle\nabla^2\overline{u}\nabla\overline{u}, n\rangle \right] d\mathcal{H}^{n-1} +$$

(3.15)

$$- \inf_{\substack{v\in H^1(\Omega) \\ v=-V_n\partial_n\overline{u} \text{ on } \Gamma_D}} \left\{ \int_\Omega |\nabla v|^2\, dx - 2\int_{\Gamma_N} v\langle\nabla\overline{u}, \nabla_\Gamma V_n\rangle\, d\mathcal{H}^{n-1} \right\}.$$

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The above equality can be established by applying Theorem 3.4 (as it continues to hold for $g(t) = -\lambda t$, *cf.* the variant discussed in §7.1). Namely in this case, as already pointed out in Remark 3.6, the second order shape derivative is given by $l_2(V_n)$, being l_2 the quadratic form defined in (3.14). The equality between $l_2(V_n)$ and the r.h.s. of (3.15) readily follows by using the identity $\Delta \overline{u} = \partial_{nn}^2 \overline{u} + \partial_n \overline{u} H_{\partial\Omega}$ on Γ_D, the fact that $\partial_n \overline{u} = 0$ on Γ_N and by taking into account that $Q_g \equiv 0$ and $Q_f(z) = \frac{|z|^2}{2}$.

We underline that in the pure Dirichlet case, *i.e.* when $\Gamma_N = \emptyset$, our formula (3.15) allows to recover the known expression of the second order shape derivative of the torsional rigidity (see for instance [25, Section 5.9.6]). Finally let us point out that, when the deformation field is not normal to the boundary, the expression of $J''(\Omega, V)$ is slightly more complicated than (3.15). Precisely, it is given by $l_1(z) + l_2(V_n)$, being l_1 and z defined respectively in (3.13) and (3.12); in other words, we have to add to the r.h.s. of formula (3.15) the following terms:

$$\int_{\Gamma_D} \frac{|\partial_n \overline{u}|^2}{2} (\langle V_\Gamma, Dn\, V_\Gamma\rangle - 2\langle V_\Gamma, \nabla_\Gamma V_n\rangle)\, d\mathcal{H}^{n-1} - \int_{\Gamma_N} \left(\frac{|\nabla \overline{u}|^2}{2} - \lambda\overline{u}\right)(\langle V_\Gamma, Dn\, V_\Gamma\rangle - 2\langle V_\Gamma, \nabla_\Gamma V_n\rangle)\, d\mathcal{H}^{n-1}.$$

Example 3.9. As a variant of the previous example, we can consider the p-torsion problem under Dirichlet boundary conditions:

$$J_p(\Omega) = -\inf\left\{\int_\Omega \left(\frac{|\nabla u|^p}{p} - \lambda u\right) dx \ : \ u \in W_0^{1,p}(\Omega)\right\}, \tag{3.16}$$

where $p > 2$ and λ is a positive parameter. Note that in this case the integrands f and g do not satisfy the standing assumptions, in particular (2.1), so that we cannot apply directly Theorem 3.4. In fact a major difficulty appears with the degeneracy at the origin of the Hessian of f. Nevertheless, by exploiting a suitable approximation argument, we are able to show that Theorem 3.4 is still valid, provided a suitable equality between weighted Sobolev spaces holds (*cf.* eq. (7.8)). The outcoming formula reads as follows. Let $\partial\Omega$ be of class C^2 and let the deformation field V be normal to the boundary. Observe that $\overline{\sigma}$ is parallel to $\nabla \overline{u}$, and satisfies $\operatorname{div}\overline{\sigma} = -\lambda$; moreover, the equalities $\langle(\operatorname{div}\overline{\sigma}I - D\overline{\sigma})n, n\rangle = \overline{\sigma}_n H_{\partial\Omega}$ and $\Delta u = \partial_{nn}^2 \overline{u} + \partial_n \overline{u} H_{\partial\Omega}$ hold on $\partial\Omega$. Taking these facts into account, starting from (3.11) we obtain:

$$J_p''(\Omega, V) = -\frac{1}{p}\int_{\partial\Omega} V_n^2 \left(p\lambda \partial_n \overline{u} + |\partial_n \overline{u}|^p H_{\partial\Omega}\right) d\mathcal{H}^{n-1} - \inf_{\substack{v \in H^1(\Omega)\\ v = -V_n\partial_n\overline{u} \text{ on } \partial\Omega}} \int_\Omega \langle P(\overline{u})\nabla v, \nabla v\rangle\, dx, \tag{3.17}$$

with

$$P(\overline{u}) := |\nabla \overline{u}|^{p-2}\left(I + (p-2)\frac{\nabla \overline{u}}{|\nabla \overline{u}|} \otimes \frac{\nabla \overline{u}}{|\nabla \overline{u}|}\right).$$

The complete presentation of this variant is postponed to §7.2, see Theorem 7.1.

4 Existence of the second order shape derivative

This section is devoted to the proof of Theorem 3.4 (i). We introduce for brevity the differential quotients

$$r_\varepsilon(V) := 2\,\frac{J(\Omega_\varepsilon) - J(\Omega) - \varepsilon J'(\Omega, V)}{\varepsilon^2}. \tag{4.1}$$

By exploiting the formulation (2.10) of $J(\Omega_\varepsilon)$ and taking therein as a test function $\overline{u} + \varepsilon w$, we infer

$$r_\varepsilon(V) \geq \frac{2}{\varepsilon^2} \left(- \int_\Omega [f_\varepsilon(\nabla \overline{u} + \varepsilon \nabla w) + g_\varepsilon(\overline{u} + \varepsilon w)] \, dx - J(\Omega, V) - \varepsilon J'(\Omega, V) \right). \tag{4.2}$$

We are thus led to introduce the sequence of functionals defined for $w \in H(\Omega)$ and $V \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ by

$$E_\varepsilon(w, V) := \frac{2}{\varepsilon^2} \left\{ \int_\Omega [f_\varepsilon(\nabla \overline{u} + \varepsilon \nabla w) + g_\varepsilon(\overline{u} + \varepsilon w)] \, dx + J(\Omega) + \varepsilon J'(\Omega, V) \right\}. \tag{4.3}$$

Moreover, still for $w \in H(\Omega)$ and $V \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, let us define the following functional $E(w, V)$, which will be seen to be the variational limit of $E_\varepsilon(w, V)$, and its "dual counterpart" $E^*(\eta, V)$ (with $\eta \in X(\Omega; \mathbb{R}^n)$ and V as above):

$$
\begin{aligned}
E(w, V) \quad := 2 \int_\Omega [f(\nabla \overline{u}) + g(\overline{u})] a_2(DV) \, dx + 2 \bigg\{ &\int_\Omega \left[Q_f(\nabla w - DV^T \nabla \overline{u}) + Q_g(w) \right] \, dx + \\
&+ \int_\Omega \left[-\langle (DV - \operatorname{div} V\, I) \nabla f(\nabla \overline{u}), \nabla w - DV^T \nabla \overline{u} \rangle \, dx + \operatorname{div} V\, g'(\overline{u}) w \right] \, dx \bigg\},
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
E^*(\eta, V) \quad := 2 \int_\Omega \left[f^*(\overline{\sigma}) + g^*(\operatorname{div} \overline{\sigma}) - \langle \nabla f^*(\overline{\sigma}), \overline{\sigma} \rangle - \overline{u} \operatorname{div} \overline{\sigma} \right] a_2(DV) \, dx + \\
+ 2 \bigg\{ \int_\Omega \left[Q_{f^*}((DV - \operatorname{div} V\, I)\overline{\sigma} + \eta) + Q_{g^*}(\operatorname{div} \eta - \operatorname{div} V \operatorname{div} \overline{\sigma}) + \langle \nabla f^*(\overline{\sigma}), DV\, \eta \rangle \right] \, dx \bigg\}.
\end{aligned}
\tag{4.5}
$$

Theorem 3.4 (i) holds in view of the following result:

Proposition 4.1. *Under the standing assumptions, the second order shape derivative $J''(\Omega, V)$ exists, and is given by*

$$J''(\Omega, V) = \lim_{\varepsilon \to 0} \left(- \inf_{w \in H(\Omega)} E_\varepsilon(w, V) \right) = - \inf_{w \in H(\Omega)} E(w, V) = \inf_{\eta \in X(\Omega; \mathbb{R}^n)} E^*(\eta, V). \tag{4.6}$$

Moreover the map $V \mapsto J''(\Omega, V)$ is a quadratic form in $C^1(\mathbb{R}^n; \mathbb{R}^n)$.

Proposition 4.1 will be obtained as a consequence of the two lemmas stated hereafter.

Lemma 4.2. (upper and lower bounds)
Under the standing assumptions, the differential quotients defined in (4.1) satisfy

$$\liminf_{\varepsilon \to 0} r_\varepsilon(V) \geq - \inf_{w \in H(\Omega)} E(w, V) \tag{4.7}$$

and

$$\limsup_{\varepsilon \to 0} r_\varepsilon(V) \leq \inf_{\eta \in X(\Omega; \mathbb{R}^n)} E^*(\eta, V). \tag{4.8}$$

Lemma 4.3. (Γ-convergence)
Under the standing assumptions, for every $V \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ the sequence $\{E_\varepsilon(\cdot, V)\}$ is equicoervice and Γ-converges to $E(\cdot, V)$, with respect to the weak topology of $H(\Omega)$.

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Let us show how the proposition follows from these lemmas, and then turn back to their proofs.

Proof of Proposition 4.1. We begin by proving the last equality in (4.6). Let $\Psi : L^2(\Omega; \mathbb{R}^n) \times L^q(\Omega) \to \mathbb{R}$ be the function

$$\Psi(z, v) := \int_\Omega \left[Q_f(z - a) + \langle z - a, b \rangle + Q_g(v) + v\alpha + \gamma \right] dx \qquad (4.9)$$

with

$$a := DV^T \nabla \overline{u}, \quad b := -(DV - \operatorname{div} V\, I)\overline{\sigma}, \quad \alpha := \operatorname{div} V g'(\overline{u}), \quad \gamma := \left[f(\nabla \overline{u}) + g(\overline{u}) \right] a_2(DV).$$

For every $w \in H(\Omega)$, it holds $E(w, V) = 2\,\Psi(\nabla w, w)$, in particular

$$-\inf_{w \in H(\Omega)} E(w, V) = -2 \inf_{w \in H(\Omega)} \left\{ \Psi(\nabla w, w) \right\}. \qquad (4.10)$$

By applying Lemma 8.5 with $Y = H(\Omega)$, $Z = L^2(\Omega; \mathbb{R}^n) \times L^q(\Omega)$, $A : Y \to Z$ the operator $Av := (\nabla v, v)$, $\Phi : Y \to \mathbb{R}$ the zero function and Ψ defined as in (4.9), we can rewrite (4.10) as

$$-2 \inf_{w \in Y} \left\{ \Psi(Aw) + \Phi(w) \right\} = 2 \inf_{(\eta, \tau) \in Z^*} \left\{ \Psi^*(\eta, \tau) + \Phi^*(-A^*(\eta, \tau)) \right\}. \qquad (4.11)$$

Let us compute the Fenchel conjugates appearing in the r.h.s. of (4.11). By exploiting Lemma 8.2, it is easy to see that

$$\Psi^*(\eta, \tau) = \int_\Omega \left[(Q_f)^*(\eta - b) + \langle \eta, a \rangle + (Q_g)^*(\tau - \alpha) + \gamma \right] dx.$$

Since $\Phi \equiv 0$, its Fenchel conjugate Φ^* is 0 at 0 and $+\infty$ otherwise. As an element of Y^*, $A^*(\eta, \tau)$ is characterized by its action on the elements of Y: since

$$\langle A^*(\eta, \tau), v \rangle_{Y^*, Y} = \langle (\eta, \tau), Av \rangle_{Z^*, Z} = \int_\Omega \left(\langle \eta, \nabla v \rangle + \tau v \right) dx,$$

we infer that $\Phi^*(A^*(\eta, \tau)) = 0$ if and only if $\tau = \operatorname{div} \eta$ (with the additional condition $\eta \cdot n = 0$ on Γ_N); note that, as a consequence, such vector fields η belong to $X(\Omega; \mathbb{R}^n)$.
Therefore we may rewrite the r.h.s. of (4.11) as

$$2 \inf_{\eta \in X(\Omega; \mathbb{R}^n)} \left\{ \int_\Omega \left[(Q_f)^*(\eta - b) + \langle \eta, a \rangle + (Q_g)^*(\operatorname{div} \eta - \alpha) + \gamma \right] dx \right\}. \qquad (4.12)$$

Since by Lemmas 8.3 and 8.4 we have $(Q_f)^* = Q_{f^*}$ and $(Q_g)^* = Q_{g^*}$, and by the optimality conditions (2.5) there holds

$$\gamma = \int_\Omega \left[f^*(\overline{\sigma}) + g^*(\operatorname{div} \overline{\sigma}) - \langle \nabla f^*(\overline{\sigma}), \overline{\sigma} \rangle - \overline{u} \operatorname{div} \overline{\sigma} \right] a_2(DV)\, dx,$$

we conclude that the expression in (4.12) agrees with $\inf_{\eta \in X(\Omega; \mathbb{R}^n)} E^*(\eta, V)$, which achieves the proof of the last equality in (4.6). Combined with Lemma 4.2, such equality implies the existence of the second order shape derivative, and the fact that it agrees with any of the two infimum problems at the r.h.s. of (4.7) and (4.8). The second equality in (4.6) is a direct consequence of Lemma 4.3.

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Finally, the fact that the map $V \mapsto J''(\Omega, V)$ is a quadratic form is readily checked: one may start from the equality $J''(\Omega, V) = -\inf_w E(w, V)$, and look at the expression (4.4) of $E(w, V)$, by recalling in particular that Q_f is a quadratic form. $\qquad\square$

Proof of Lemma 4.2. We first prove the lower bound in (4.7). We start from the inequality (4.2), where we choose w as an element of $C^\infty(\overline{\Omega}) \cap H(\Omega)$. Recalling the definitions (2.11) of f_ε and g_ε, the definition (1.1) of $J(\Omega)$, and the expression (2.8) of $J'(\Omega; V)$, we can rewrite the inequality (4.2) separating the terms of different orders in ε. Recalling that M_ε and m_ε are defined respectively according to (2.15) and (2.16), and setting for brevity

$$h_1 := \nabla w - DV^T \nabla \overline{u}, \quad h_2 := (DV^T)^2 \nabla \overline{u} - DV^T \nabla w, \quad h_\varepsilon := (DV^T)^2 \nabla w + M_\varepsilon(\nabla \overline{u} + \varepsilon \nabla w),$$

we get

$$r_\varepsilon(V) \geq \Big(I_0(\varepsilon) + I_1(\varepsilon) + I_2(\varepsilon)\Big), \tag{4.13}$$

where

$$I_0(\varepsilon) := -2 \int_\Omega [f(\nabla \overline{u} + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_\varepsilon) + g(\overline{u} + \varepsilon w)](a_2(DV) + \varepsilon m_\varepsilon)\, dx \,,$$

$$I_1(\varepsilon) := -\frac{2}{\varepsilon} \int_\Omega [f(\nabla \overline{u} + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_\varepsilon) + g(\overline{u} + \varepsilon w) - f(\nabla \overline{u}) - g(\overline{u})]\, \mathrm{div}\, V\, dx \,,$$

$$I_2(\varepsilon) := -\frac{2}{\varepsilon^2} \int_\Omega [f(\nabla \overline{u} + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_\varepsilon) + g(\overline{u} + \varepsilon w) - f(\nabla \overline{u}) - g(\overline{u}) + \varepsilon \langle \nabla f(\nabla \overline{u}), DV^T \nabla \overline{u} \rangle]\, dx \,.$$

Let us study separately the asymptotic behavior of $I_i(\varepsilon)$ as $\varepsilon \to 0$. By exploiting the growth assumptions on the integrands and the uniform L^∞ boundedness of m_ε and h_ε, we get

$$\lim_{\varepsilon \to 0} I_0(\varepsilon) = -2 \int_\Omega [f(\nabla \overline{u}) + g(\overline{u})] a_2(DV)\, dx \,. \tag{4.14}$$

By applying Lemma 8.1 (i) to the integral functionals I_f and I_g defined according to (8.1) (note that this can be done after possibly modifying g outside an interval, since $\overline{u} + \varepsilon w$ remains uniformly bounded in L^∞), we get

$$\lim_{\varepsilon \to 0} I_1(\varepsilon) = -2 \int_\Omega [\langle \nabla f(\nabla \overline{u}), \nabla w - DV^T \nabla \overline{u} \rangle + g'(\overline{u})w]\, \mathrm{div}\, V\, dx \,. \tag{4.15}$$

Let us now consider $I_2(\varepsilon)$. We recall that, in view of the optimality conditions in (2.5), there holds $\overline{\sigma} = \nabla f(\nabla \overline{u})$ and $\mathrm{div}\, \overline{\sigma} = g'(\overline{u})$, so that

$$\int_\Omega [\langle \nabla f(\nabla \overline{u}), \nabla w \rangle + g'(\overline{u})w]\, dx = \int_\Omega \big(\langle \overline{\sigma}, \nabla w \rangle + w\, \mathrm{div}\, \overline{\sigma}\big)\, dx = \int_{\partial\Omega} w \overline{\sigma} \cdot n\, d\mathcal{H}^{n-1} = 0 \,, \tag{4.16}$$

where the last equality holds since $w = 0$ on Γ_D and $\overline{\sigma} \cdot n = 0$ on Γ_N. Thus, $I_2(\varepsilon)$ remains unchanged if we add the zero term (4.16) (multiplied by $2/\varepsilon$):

$$I_2(\varepsilon) = -\frac{2}{\varepsilon^2} \int_\Omega [f(\nabla \overline{u} + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_\varepsilon) - f(\nabla \overline{u}) - \varepsilon \langle \nabla f(\nabla u), h_1 \rangle]\, dx +$$

$$-\frac{2}{\varepsilon^2} \int_\Omega [g(\overline{u} + \varepsilon w) - g(\overline{u}) - \varepsilon g'(\overline{u})w]\, dx \,.$$

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Hence, by applying Lemma 8.1, we obtain

$$\lim_{\varepsilon \to 0} I_2(\varepsilon) = -2 \int_{\Omega} [Q_f(h_1) + \langle \nabla f(\nabla \overline{u}), h_2 \rangle + Q_g(w)] \, dx \,. \tag{4.17}$$

By combining (4.13), (4.14), (4.15) and (4.17), by the arbitrariness of $w \in C^{\infty}(\overline{\Omega}) \cap H(\Omega)$ we infer

$$\liminf_{\varepsilon \to 0} r_{\varepsilon}(V) \geq - \inf_{w \in C^{\infty}(\overline{\Omega}) \cap H(\Omega)} E(w, V) \,. \tag{4.18}$$

In order to conclude the proof of the inequality (4.7), it is enough to observe that the infimum at the r.h.s. of (4.18) coincides with the infimum at the r.h.s. of (4.7). Indeed, the integral functional $E(\cdot, V)$ above is continuous on $H(\Omega)$ (by the growth conditions (2.2) and the Sobolev embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$), whereas $C^{\infty}(\overline{\Omega}) \cap H(\Omega)$ is dense in $H(\Omega)$.

Let us now prove the upper bound in (4.8). Let η be an arbitrary element of the space $C^{\infty}(\overline{\Omega}; \mathbb{R}^n) \cap X(\Omega; \mathbb{R}^n)$. In view of (1.2), the differential quotients $r_{\varepsilon}(V)$ introduced in (4.1) can be rewritten as

$$r_{\varepsilon}(V) = \frac{2}{\varepsilon^2} \left[J^*(\Omega_{\varepsilon}) - J^*(\Omega) - \varepsilon J'(\Omega, V) \right] \,.$$

Then, by exploiting the expression of $J^*(\Omega_{\varepsilon})$ in (2.13), we infer that

$$r_{\varepsilon}(V) \leq \frac{2}{\varepsilon^2} \left(\int_{\Omega} [f_{\varepsilon}^*(\overline{\sigma} + \varepsilon \eta) + g_{\varepsilon}^*(\operatorname{div} \overline{\sigma} + \varepsilon \operatorname{div} \eta)] \, dx - J^*(\Omega) - \varepsilon J'(\Omega, V) \right) \,. \tag{4.19}$$

Recalling the definitions (2.14) of f_{ε}^* and g_{ε}^*, the definition of $J^*(\Omega)$ in (1.2), and the second expression for $J'(\Omega; V)$ in (2.8), we can rewrite the inequality (4.19) separating the terms of different orders in ε. Recalling the definition (2.17) of $\widetilde{m_{\varepsilon}}$, and setting for brevity

$$
\begin{aligned}
z_1 :=& (DV - \operatorname{div} V I)\overline{\sigma} + \eta \,, \\
z_2 :=& (DV - \operatorname{div} V I)\eta + ((\operatorname{div} V)^2 - \operatorname{div} V \, DV - a_2(DV))\overline{\sigma} \,, \\
z_{\varepsilon} :=& \widetilde{m_{\varepsilon}}(\overline{\sigma} + \varepsilon \eta) + ((\operatorname{div} V)^2 - a_2(DV))(DV\overline{\sigma} + \eta) - \operatorname{div} V \, DV \eta \,, \\
\tau_1 :=& \operatorname{div} \eta - \operatorname{div} V \operatorname{div} \overline{\sigma} \,, \\
\tau_2 :=& ((\operatorname{div} V)^2 - a_2(DV)) \operatorname{div} \overline{\sigma} - \operatorname{div} V \operatorname{div} \eta \,, \\
\tau_{\varepsilon} :=& ((\operatorname{div} V)^2 - a_2(DV)) \operatorname{div} \eta + \widetilde{m_{\varepsilon}}(\operatorname{div} \overline{\sigma} + \varepsilon \operatorname{div} \eta) \,,
\end{aligned}
$$

we obtain

$$r_{\varepsilon}(V) \leq \left(I_0^*(\varepsilon) + I_1^*(\varepsilon) + I_2^*(\varepsilon) \right) \,, \tag{4.20}$$

where

$$I_0^*(\varepsilon) := 2 \int_{\Omega} [f^*(\overline{\sigma} + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_{\varepsilon}) + g^*(\overline{\sigma} + \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \varepsilon^3 \tau_{\varepsilon})](a_2(DV) + \varepsilon m_{\varepsilon}) \, dx \,,$$

$$I_1^*(\varepsilon) := \frac{2}{\varepsilon} \int_{\Omega} [f^*(\overline{\sigma} + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_{\varepsilon}) + g^*(\overline{\sigma} + \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \varepsilon^3 \tau_{\varepsilon}) - f^*(\overline{\sigma}) - g^*(\operatorname{div} \overline{\sigma})] \operatorname{div} V \, dx \,,$$

$$\begin{aligned} I_2^*(\varepsilon) := \frac{2}{\varepsilon^2} \int_{\Omega} [&f^*(\overline{\sigma} + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_{\varepsilon}) + g^*(\overline{\sigma} + \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \varepsilon^3 \tau_{\varepsilon}) - f^*(\overline{\sigma}) - g^*(\operatorname{div} \overline{\sigma}) + \\ &- \varepsilon \langle \nabla f^*(\overline{\sigma}), (DV - \operatorname{div} V)\overline{\sigma} \rangle + \varepsilon \operatorname{div} V \operatorname{div} \overline{\sigma}(g^*)'(\operatorname{div} \overline{\sigma})] \, dx \,. \end{aligned}$$

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Let us study separately the asymptotic behavior of $I_i^*(\varepsilon)$ as $\varepsilon \to 0$. By exploiting the growth properties of the integrands (note that the growth assumptions on f and g made in (2.2) imply similar conditions on f^* and g^*) and the uniform L^∞ boundedness of $\widetilde{m_\varepsilon}$, we get

$$\lim_{\varepsilon \to 0} I_0^*(\varepsilon) = 2 \int_\Omega [f^*(\overline{\sigma}) + g^*(\operatorname{div}\overline{\sigma})] a_2(DV) \, dx \,. \tag{4.21}$$

By applying Lemma 8.1 (i) to the integral functionals I_{f^*} and I_{g^*} defined according to (8.1) (note that this can be done after possibly modifying g^* outside an interval, since $\operatorname{div}\overline{\sigma} + \varepsilon \operatorname{div}\eta$ remains uniformly bounded in L^∞), we get

$$\lim_{\varepsilon \to 0} I_1^*(\varepsilon) = 2 \int_\Omega [\langle \nabla f^*(\overline{\sigma}), z_1 \rangle + (g^*)'(\operatorname{div}\overline{\sigma})\tau_1] \operatorname{div}V \, dx \,. \tag{4.22}$$

Let us now consider $I_2^*(\varepsilon)$. We recall that, in view of the optimality conditions (2.5), we have

$$\int_\Omega [\langle \nabla f^*(\overline{\sigma}), \eta \rangle + (g^*)'(\operatorname{div}\overline{\sigma}) \operatorname{div}\eta] \, dx = 0 \,. \tag{4.23}$$

Thus, $I_2^*(\varepsilon)$ remains unchanged by adding the zero term (4.23) (multiplied by $-2/\varepsilon$):

$$I_2^*(\varepsilon) = \frac{2}{\varepsilon^2} \int_\Omega [f^*(\overline{\sigma} + \varepsilon z_1 + \varepsilon^2 z_2 + \varepsilon^3 z_\varepsilon) - f^*(\overline{\sigma}) - \varepsilon \langle \nabla f^*(\overline{\sigma}), z_1 \rangle +$$
$$+ \, g^*(\overline{\sigma} + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \varepsilon^3\tau_\varepsilon) - g^*(\operatorname{div}\overline{\sigma}) - \varepsilon(g^*)'(\operatorname{div}\overline{\sigma})\tau_1] \, dx \,.$$

Hence, by applying Lemma 8.1, we infer

$$\lim_{\varepsilon \to 0} I_2^*(\varepsilon) = 2 \int_\Omega [Q_{f^*}(z_1) + \langle \nabla f^*(\overline{\sigma}), z_2 \rangle + Q_{g^*}(\tau_1) + (g^*)'(\operatorname{div}\overline{\sigma})\tau_2] \, dx \,. \tag{4.24}$$

By combining (4.20) with (4.21), (4.22) and (4.24) and recalling the definition (4.5) of E^*, in view of the arbitrariness of $\eta \in C^\infty(\overline{\Omega};\mathbb{R}^n) \cap X(\Omega;\mathbb{R}^n)$, we infer

$$\limsup_{\varepsilon \to 0} r_\varepsilon(V) \leq \inf_{\eta \in C^\infty(\overline{\Omega};\mathbb{R}^n) \cap X(\Omega;\mathbb{R}^n)} E^*(\eta, V) \,. \tag{4.25}$$

Finally we observe that, for every $\eta \in X(\Omega;\mathbb{R}^n)$, there exists a sequence $\{\eta_h\} \subset C^\infty(\overline{\Omega};\mathbb{R}^n) \cap X(\Omega;\mathbb{R}^n)$, such that $\eta_h \to \eta$ in $L^2(\Omega;\mathbb{R}^n)$ and $\operatorname{div}\eta_h \to \operatorname{div}\eta$ in $L^{q'}(\Omega)$. Since $E^*(\cdot, V)$ is continuous with respect to such convergence (thanks to the fact that the growth assumptions on f and g made in (2.2) imply similar conditions on f^* and g^*), the infimum at the r.h.s. of (4.25) turns out to agree with the infimum at the r.h.s. of (4.8), and the proof is achieved.
$\square$

Proof of Lemma 4.3. For brevity, throughout the proof we fix the vector field V and we omit it when writing the functionals E_ε and E. We begin by showing the equicoercivity of E_ε. In view of the convexity assumptions on f and g we infer that the following bounds hold true, where m and k are the positive constants appearing in (2.1):

$$f(z_0 + z) \geq f(z_0) + \langle \nabla f(z_0), z \rangle + m|z|^2 \,, \qquad g(v_0 + v) \geq g(v_0) + g'(v_0)v + k|v|^2 \,.$$

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By exploiting such bounds for $z_0 = \nabla \overline{u}$, $z = -\nabla \overline{u} + D\Psi_\varepsilon^{-T}(\nabla \overline{u} + \nabla w)$, $v_0 = \overline{u}$ and $v = \varepsilon w$, we infer that for every ε

$$E_\varepsilon(w) \geq c_1 \|\nabla w\|_{L^2}^2 + c_2 \|w\|_{L^2}^2 - c_3 \|\nabla w\|_{L^2} - c_4 \|w\|_{L^2} - c_5,$$

for some positive constants c_i. Therefore the sequence E_ε is equicoercive in $H(\Omega)$.

Let us now show the Γ-convergence statement. By definition of Γ-convergence, we have to prove the so-called Γ-liminf and Γ-limsup inequalities:

$$\inf \left\{ \liminf E_\varepsilon(w_\varepsilon) \ : \ v_\varepsilon \overset{H(\Omega)}{\rightharpoonup} v \right\} \geq E(w) \tag{4.26}$$

$$\inf \left\{ \limsup E_\varepsilon(w_\varepsilon) \ : \ v_\varepsilon \overset{H(\Omega)}{\rightharpoonup} v \right\} \leq E(w). \tag{4.27}$$

Let us prove (4.26). For every $w \in H(\Omega)$ and for every sequence w_ε which converges weakly to w in $H(\Omega)$, we have to prove that

$$E(w) \leq \liminf_\varepsilon E_\varepsilon(w_\varepsilon). \tag{4.28}$$

For a given $w \in H(\Omega)$, let us first consider a regular sequence $\widetilde{w}_\varepsilon \in C^\infty(\overline{\Omega}) \cap H(\Omega)$ weakly converging to w.

As already done in the proof of Lemma 4.2, by exploiting the C^1 regularity of the integral functionals I_f and I_g and the L^2-Mosco convergence of the corresponding sequences $\Delta_{\varepsilon,f}$ and $\Delta_{\varepsilon,g}$ defined according to (8.2), we infer

$$E(w) \leq \liminf_\varepsilon E_\varepsilon(\widetilde{w}_\varepsilon). \tag{4.29}$$

Let us now consider a generic sequence $w_\varepsilon \in H(\Omega)$ weakly converging to w. By density, for every fixed ε, there exists $\widetilde{w}_{\varepsilon,k} \in C^\infty(\overline{\Omega}) \cap H(\Omega)$ such that $\widetilde{w}_{\varepsilon,k} \to w_\varepsilon$ strongly in $H(\Omega)$ as $k \to +\infty$. With a diagonal argument, we may find a subsequence such that

$$\liminf_\varepsilon E_\varepsilon(w_\varepsilon) = \liminf_\varepsilon \lim_k E_\varepsilon(\widetilde{w}_{\varepsilon,k}) = \lim_\varepsilon E_\varepsilon(\widetilde{w}_{\varepsilon,k_\varepsilon}), \tag{4.30}$$

where the first equality follows by the strong continuity of E_ε. We remark that the subsequence $\widetilde{w}_{\varepsilon,k_\varepsilon}$ is regular and weakly converging to w, hence it satisfies (4.29). Such property, combined with the equality (4.30), concludes the proof of (4.28).

Let us now prove (4.27). For every fixed $w \in H(\Omega)$ we have to find a recovery sequence, namely a sequence w_ε which converges weakly to w in $H(\Omega)$ and satisfies $\limsup_{\varepsilon \to 0} E_\varepsilon(w_\varepsilon) \leq E(w)$. If $\widetilde{w}$ is an element of the space $C^\infty(\overline{\Omega}) \cap H(\Omega)$, we are done simply by taking the constant sequence $w_\varepsilon \equiv \widetilde{w}$. Indeed, by following the same procedure adopted in the proof of Lemma 4.2, we infer that $\lim_{\varepsilon \to 0} E_\varepsilon(\widetilde{w}) = E(\widetilde{w})$. If w is a generic element of $H(\Omega)$, we approximate it by a sequence $\widetilde{w_k} \in C^\infty(\overline{\Omega}) \cap H(\Omega)$: by the lower semicontinuity of the l.h.s. of (4.27) (usually called Γ-$\limsup E_\varepsilon$) and the continuity of E, we obtain

$$(\Gamma - \limsup E_\varepsilon)(w) \leq \liminf_k (\Gamma - \limsup E_\varepsilon)(\widetilde{w_k}) \leq \liminf_k \limsup_\varepsilon E_\varepsilon(\widetilde{w_k}) = E(w).$$

$$\square$$

5 Regularity of the solution and necessary optimality condition

In this section we prove Propositions 3.2 and 3.3. Both of them are based on the crucial result given below, which in turn exploits the Γ-convergence statement given in the previous section (*cf.* Lemma 4.3).

Proposition 5.1. *Under the standing assumptions, if V has compact support contained into Ω, then the function $\overline{\theta}_V := \langle V, \nabla \overline{u} \rangle$ solves the minimization problem $\inf\limits_{w \in H(\Omega)} E(\cdot, V)$, where $E(\cdot, V)$ is the functional defined by (4.4).*

Proof. We first prove the following claim: for every fixed $\varepsilon > 0$, the function

$$w_\varepsilon(x) := \frac{\overline{u}(x + \varepsilon V(x)) - \overline{u}(x)}{\varepsilon} \tag{5.1}$$

solves the minimization problem $\inf\limits_{w \in H(\Omega)} E_\varepsilon(\cdot, V)$, where $E_\varepsilon(\cdot, V)$ is the functional defined by (4.3).

Let $u_\varepsilon \in H(\Omega)$ be optimal for problem $J(\Omega_\varepsilon)$ in the formulation (2.10) set on the fixed domain Ω. Via change of variables we obtain

$$J(\Omega_\varepsilon) = -\int_\Omega [f_\varepsilon(\nabla u_\varepsilon) + g_\varepsilon(u_\varepsilon)] \, dx = -\int_{\Omega_\varepsilon} [f(\nabla(u_\varepsilon \circ \Psi_\varepsilon^{-1})) + g(u_\varepsilon \circ \Psi_\varepsilon^{-1})] \, dx \,.$$

We infer that the function $u_\varepsilon \circ \Psi_\varepsilon^{-1} \in H(\Omega_\varepsilon)$ is optimal for problem $J(\Omega_\varepsilon)$ in its original formulation as an infimum over $H(\Omega_\varepsilon)$. On the other hand, since V has compact support contained into Ω, we know that $J(\Omega_\varepsilon) = J(\Omega)$, and hence (since $\overline{u}$ is the *unique* solution to $J(\Omega)$) we have $u_\varepsilon \circ \Psi_\varepsilon^{-1} = \overline{u}$, namely $u_\varepsilon(x) = \overline{u} \circ \Psi_\varepsilon(x) = \overline{u}(x + \varepsilon V(x))$. It follows that the function w_ε defined in (5.1) minimizes $E_\varepsilon(\cdot, V)$ over $H(\Omega)$, and our claim on the optimality of w_ε is proved.

In view of Lemma 4.3, the minimizing sequence $\{w_\varepsilon\}$ is equibounded in $H(\Omega)$ and, up to subsequences, it weakly converges to a minimizer of the Γ-limit, namely to a minimizer of $E(\cdot, V)$. Clearly, the weak limit of w_ε necessarily agrees with the pointwise limit $\langle V, \nabla \overline{u} \rangle$, which allows to conclude that $\langle V, \nabla \overline{u} \rangle$ minimizes $E(\cdot, V)$ over $H(\Omega)$. $\qquad\square$

Proof of Proposition 3.2. In view of Proposition 5.1 we can assert that, for every vector field V compactly supported into Ω, the function $\langle V, \nabla \overline{u} \rangle$ belongs to $H^1(\Omega)$. By the arbitrariness of $V \in C_0^1(\Omega; \mathbb{R}^n)$, we infer that $\overline{u}$ is an element of $W_{\mathrm{loc}}^{2,2}(\Omega)$. $\qquad\square$

Proof of Proposition 3.3. We consider the minimization of the functional $E(\cdot, V)$ in (4.4) over $H(\Omega)$, written as done in the proof of Proposition 4.1, see eq. (4.10). By duality, we have the equality (4.11). Then, by Lemma 8.5, a function $w \in H(\Omega)$ and a pair $(\eta, \tau) \in L^2(\Omega; \mathbb{R}^n) \times L^{q'}(\Omega)$ are optimal respectively for the two problems in (4.11) if and only if $(\eta, \tau) \in \partial\Psi(\nabla w, w)$ and $(\mathrm{div}\, \eta, \tau) \in \partial\Phi(w)$. Then, by computing explicitly the expressions of $\partial\Phi$ and $\partial\Psi$, we get $\tau = \mathrm{div}\, \eta$, with the additional condition $\eta \cdot n = 0$ on Γ_N, and

$$\begin{cases} \eta & = \nabla^2 f(\nabla \overline{u})(\nabla w - DV^T \nabla \overline{u}) - (DV - \mathrm{div}\, V\, I)\overline{\sigma} \,, \\[2mm] \mathrm{div}\, \eta & = g''(\overline{u})w + g'(\overline{u}) \, \mathrm{div}\, V \,. \end{cases}$$

Hence, for $V \in C_0^1(\Omega; \mathbb{R}^n)$, the distributional equality (3.3) follows by replacing w by $\langle V, \nabla \overline{u} \rangle$ in the above optimality conditions (since we know by Proposition 5.1 that $\overline{\theta}_V = \langle V, \nabla \overline{u} \rangle$ minimizes $E(\cdot, V)$ over $H(\Omega)$). For $V \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, the equality (3.3) continues to hold by the following simple argument: in order to check (3.3) for a given test function $\varphi \in \mathcal{D}(\Omega)$, it is enough to use its validity when V is replaced by the product $\mu V \in C_0^1(\Omega; \mathbb{R}^n)$, being μ a cut off function compactly supported into Ω which equals 1 on the support of φ.

$\square$

6 Representation formula for the second order shape derivative

In this section we give the

Proof of Theorem 3.4 (ii). We start from the following equality that we have already proved in Proposition 4.1:

$$J''(\Omega, V) = - \inf_{w \in H(\Omega)} E(w, V). \tag{6.1}$$

Throughout the proof we set for brevity $\overline{\theta}_V := \langle V, \nabla \overline{u} \rangle$. By inserting into the definition (4.4) of $E(w, V)$ the identities

$$Q_f(\nabla w - DV^T \nabla \overline{u}) = Q_f\big(\nabla(w - \overline{\theta}_V)\big) + \big\langle \nabla^2 f(\nabla \overline{u})(\nabla^2 \overline{u})V, \nabla(w - \overline{\theta}_V) \big\rangle + Q_f\big((\nabla^2 \overline{u})V\big)$$

$$Q_g(w) = Q_g(w - \overline{\theta}_V) + g''(\overline{u}) \overline{\theta}_V (w - \overline{\theta}_V) + Q_g(\overline{\theta}_V),$$

we obtain

$$E(w, V) = \int_\Omega 2\big[Q_f\big(\nabla(w - \overline{\theta}_V)\big) + Q_g(w - \overline{\theta}_V)\big]\, dx +$$

$$+ \int_\Omega 2\big[\langle \nabla^2 f(\nabla \overline{u})(\nabla^2 \overline{u})V - (DV - \operatorname{div} V I)\overline{\sigma}, \nabla(w - \overline{\theta}_V) \rangle + \big(g''(\overline{u}) \overline{\theta}_V + \operatorname{div} V g'(\overline{u})\big)(w - \overline{\theta}_V)\big]\, dx +$$

$$+ \int_\Omega \big[\big(f(\nabla \overline{u}) + g(\overline{u})\big)(\operatorname{div} V I - DV) : DV^T - 2\langle(DV - \operatorname{div} V I)\overline{\sigma}, (\nabla^2 \overline{u})V \rangle + 2 \operatorname{div} V g'(\overline{u})\overline{\theta}_V\big]\, dx +$$

$$+ \int_\Omega \big[2Q_f((\nabla^2 \overline{u})V) + 2Q_g(\overline{\theta}_V)\big]\, dx.$$

Recalling the definition (3.5) of the quadratic form $\mathcal{Q}(\overline{u}, \cdot)$ and the definition (3.2) of the vector field $B(\overline{u}, V)$, by using also Proposition 3.3 we can rewrite the above equality as $E(w, V) = (\mathrm{I}) + (\mathrm{II})$, with

$$(\mathrm{I}) = \mathcal{Q}(\overline{u}, w - \overline{\theta}_V)\, dx + \int_\Omega 2\big[\langle B(\overline{u}, V), \nabla(w - \overline{\theta}_V) \rangle + \operatorname{div} B(\overline{u}, V)(w - \overline{\theta}_V)\big]\, dx$$

and

$$(\mathrm{II}) = \int_\Omega \big[\big(f(\nabla \overline{u}) + g(\overline{u})\big)(\operatorname{div} V I - DV) : DV^T - 2\langle(DV - \operatorname{div} V I)\overline{\sigma}, (\nabla^2 \overline{u})V \rangle + 2 \operatorname{div} V g'(\overline{u})\overline{\theta}_V\big]\, dx +$$

$$+ \int_\Omega \big[2Q_f((\nabla^2 \overline{u})V) + 2Q_g(\overline{\theta}_V)\big]\, dx.$$

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Note that, in view of the assumption $\overline{u} \in W^{2,2}(\Omega)$, the vector field $B(\overline{u}, V)$ is in L^2 and has bounded divergence, thus it admits a normal trace in $H^{-1/2}(\partial\Omega)$, and we are allowed to apply the integration by parts formula (2.3). Thus we get

$$\text{(I)} = \mathcal{Q}(\overline{u}, w - \overline{\theta}_V)\, dx + \int_{\partial\Omega} 2(w - \overline{\theta}_V)\, B(\overline{u}, V) \cdot n\, d\mathcal{H}^{n-1}, \tag{6.2}$$

We now want to rewrite (II) as a boundary integral. To this purpose we observe that, by (3.2) and (3.3), there hold

$$-\langle (DV - \operatorname{div} V I)\overline{\sigma} - \nabla^2 f(\nabla^2 \overline{u})V, (\nabla^2 \overline{u})V \rangle = \langle B(\overline{u}, V), (\nabla^2 \overline{u})V \rangle, \tag{6.3}$$

$$\operatorname{div} V g'(\overline{u})\, \overline{\theta}_V + g''(\overline{u})(\overline{\theta}_V)^2 = \operatorname{div} B(\overline{u}, V)\, \overline{\theta}_V. \tag{6.4}$$

Moreover, again thanks to the assumption $\overline{u} \in W^{2,2}(\Omega)$, the vector field

$$X_1 := (f(\nabla \overline{u}) + g(\overline{u}))(\operatorname{div} V I - DV)V \tag{6.5}$$

is bounded and has L^2 divergence. Thus we are allowed to apply the integration by parts formula (2.3) and we get

$$\int_\Omega (f(\nabla \overline{u}) + g(\overline{u}))(\operatorname{div} V I - DV) : DV^T\, dx = \int_{\partial\Omega} (f(\nabla \overline{u}) + g(\overline{u}))\langle (\operatorname{div} V I - DV)V, n\rangle\, d\mathcal{H}^{n-1} +$$

$$- \int_\Omega \langle (\operatorname{div} V I - DV)V, (\nabla^2 \overline{u})\overline{\sigma} + g'(\overline{u})\nabla \overline{u}\rangle\, dx. \tag{6.6}$$

Using (6.3), (6.4) and (6.6), we infer that

$$\text{(II)} = \int_{\partial\Omega} (X_1 \cdot n)\, d\mathcal{H}^{n-1} + \int_\Omega [\langle B(\overline{u}, V), (\nabla^2 \overline{u})V\rangle + \operatorname{div} B(\overline{u}, V)\, \overline{\theta}_V]\, dx +$$

$$- \int_\Omega \langle (\nabla^2 \overline{u})DV\overline{\sigma} - DV^T(\nabla^2 \overline{u})\overline{\sigma} - g'(\overline{u})DV^T\nabla \overline{u}, V\rangle\, dx.$$

Exploiting the equality $(\nabla^2 \overline{u})V = \nabla \overline{\theta}_V - DV^T\nabla \overline{u}$, we get

$$\text{(II)} = \int_{\partial\Omega} (X_1 \cdot n)\, d\mathcal{H}^{n-1} + \int_\Omega [\langle B(\overline{u}, V), \nabla \overline{\theta}_V\rangle + \operatorname{div} B(\overline{u}, V)\, \overline{\theta}_V]\, dx +$$

$$- \int_\Omega \left[\langle \nabla^2 f(\nabla \overline{u})(\nabla^2 \overline{u})V, DV^T\nabla \overline{u}\rangle + \operatorname{div} V \langle \overline{\sigma}, DV^T\nabla \overline{u}\rangle + \langle (\nabla^2 \overline{u})DV\overline{\sigma}, V\rangle \right]\, dx +$$

$$+ \int_\Omega \left[\langle DV\overline{\sigma}, DV^T\nabla \overline{u}\rangle + \langle DV^T(\nabla^2 \overline{u})\overline{\sigma}, V\rangle + g'(\overline{u})\langle DV^T\nabla \overline{u}, V\rangle \right]\, dx.$$

Finally we remark that $\langle B(\overline{u}, V), \nabla \overline{\theta}_V\rangle + \operatorname{div} B(\overline{u}, V)\, \overline{\theta}_V = \operatorname{div}\big(\overline{\theta}_V B(\overline{u}, V)\big)$, and

$$- \langle \nabla^2 f(\nabla \overline{u})(\nabla^2 \overline{u})V, DV^T\nabla \overline{u}\rangle - \operatorname{div} V \langle \overline{\sigma}, DV^T\nabla \overline{u}\rangle - \langle (\nabla^2 \overline{u})DV\overline{\sigma}, V\rangle + \langle DV\overline{\sigma}, DV^T\nabla \overline{u}\rangle +$$

$$+ \langle DV^T(\nabla^2 \overline{u})\overline{\sigma}, V\rangle + g'(\overline{u})\langle DV^T\nabla \overline{u}, V\rangle = \operatorname{div}\left(-\langle DV\overline{\sigma}, \nabla \overline{u}\rangle V + \langle DV V, \nabla \overline{u}\rangle \overline{\sigma}\right).$$

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Hence

$$(\text{II}) = \int_{\partial\Omega} (X_1 \cdot n)\, d\mathcal{H}^{n-1} + \int_\Omega \operatorname{div}\left(\overline{\theta}_V\, B(\overline{u}, V) - \langle DV\overline{\sigma}, \nabla\overline{u}\rangle V + \langle DV\, V, \nabla\overline{u}\rangle\overline{\sigma}\right) dx\,; \qquad (6.7)$$

moreover, thanks to the assumption $\overline{u} \in W^{2,2}(\Omega)$, the tensor field

$$X_2 := \overline{\theta}_V\, B(\overline{u}, V) - \langle DV\overline{\sigma}, \nabla\overline{u}\rangle V + \langle DV\, V, \nabla\overline{u}\rangle\overline{\sigma}\,, \qquad (6.8)$$

is in L^2 and has L^2 divergence. Then, by adding (6.2) and (6.7) and by applying the integration by parts formula (2.3), we obtain

$$
\begin{aligned}
E(w, V) = \quad & \mathcal{Q}(\overline{u}, w - \overline{\theta}_V)\, dx + \int_{\partial\Omega} 2(w - \overline{\theta}_V)\, B(\overline{u}, V) \cdot n\, d\mathcal{H}^{n-1} + \\
& + \int_{\partial\Omega} (X_1 \cdot n)\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} (X_2 \cdot n)\, d\mathcal{H}^{n-1}\,.
\end{aligned}
\qquad (6.9)
$$

Now, the fact that the field $\mathcal{C}(\overline{u}, V)$ defined in (3.4) admits a normal trace in $H^{-1/2}(\partial\Omega)$ follows by observing that $\mathcal{C}(\overline{u}, V) = -X_1 - X_2$, with X_1 and X_2 defined respectively in (6.5) and (6.8). Finally the expression (3.6) of the second order shape derivative follows by combining (6.9) with (6.1), and using the equality $\mathcal{C}(\overline{u}, V) = -X_1 - X_2$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7 Variants and perspectives

7.1 The case g linear

We claim that Theorem 3.4, Proposition 3.2, and Proposition 3.3 remain true in the case when $g(v) = -\lambda v$, for some $\lambda \in \mathbb{R}$. Note just that in the definition (3.5) of the quadratic form $\mathcal{Q}(\overline{u}, \cdot)$, the term Q_g vanishes.

The proof of the lower bound inequality (4.7) works unaltered, whereas the proof of the upper bound inequality (4.8) must be modified as follows. Note firstly that, since the Fenchel conjugate of g is the Dirac delta function $g^*(\tau) = \delta_{-\lambda}$, the dual energy $E^*(\eta, V)$ introduced in (4.5) is finite only if $\operatorname{div}\eta = -\lambda\operatorname{div}V$, and in this case it reads:

$$
\begin{aligned}
E^*(\eta, V) = \quad & 2\int_\Omega [f^*(\overline{\sigma}) - \langle \nabla f^*(\overline{\sigma}), \overline{\sigma}\rangle + \lambda\overline{u}]\, a_2(DV)\, dx + \\
& + 2\int_\Omega \left[Q_{f^*}((DV - \operatorname{div}V\, I)\overline{\sigma} + \eta) + \langle\nabla f^*(\overline{\sigma}), DV\,\eta\rangle\right] dx\,.
\end{aligned}
\qquad (7.1)
$$

Next, we write the following inequality (which replaces (4.19)):

$$r_\varepsilon(V) \le \frac{2}{\varepsilon^2}\left(\int_\Omega [f_\varepsilon^*(\sigma_\varepsilon) + g_\varepsilon^*(\operatorname{div}\sigma_\varepsilon)]\, dx - J^*(\Omega, V) - \varepsilon J'(\Omega, V)\right), \qquad (7.2)$$

where σ_ε are the perturbations of $\overline{\sigma}$ defined by

$$\sigma_\varepsilon := \overline{\sigma} + \sum_{k=1}^{n} \varepsilon^k \eta_k\,, \qquad (7.3)$$

with for $k \in \{1, 2, \ldots, n\}$

$$\eta_k \in L^\infty(\Omega; \mathbb{R}^n), \qquad \operatorname{div} \eta_k = -\lambda a_k(DV) \ \text{ in } \Omega, \qquad \eta_k \cdot n = 0 \text{ on } \Gamma_N. \tag{7.4}$$

Notice that, for such a choice of σ_ε, the r.h.s. of the inequality (7.2) is finite. Namely, exploiting the expansion (2.12) and the fact that by (2.5) it holds $\operatorname{div} \overline{\sigma} = -\lambda$, we obtain $\operatorname{div} \sigma_\varepsilon = -\lambda \beta_\varepsilon$, so that

$$g_\varepsilon^*(\operatorname{div} \sigma_\varepsilon) = \beta_\varepsilon g^*(\beta_\varepsilon^{-1} \operatorname{div} \sigma_\varepsilon) = \beta_\varepsilon g^*(-\lambda) = 0. \tag{7.5}$$

Notice also that the set of fields satisfying (7.4) is non empty: indeed, every $a_k(DV)$ can be expressed as the divergence of a suitable regular vector field ν_k (see for instance [33, Lemma 4.6.4]); thus the conditions in (7.4) are verified by taking $\eta_k := -\lambda \nu_k + \nabla w_k$, being w_k the solution of

$$\Delta w_k = 0 \quad \text{in } \Omega, \qquad \partial_n w_k = h_k \quad \text{on } \partial\Omega,$$

with $h_k = \lambda \nu_k \cdot n$ on Γ_N, $\int_{\partial\Omega} h_k = 0$. By combining (7.2) and (7.5), we may write

$$r_\varepsilon(V) \le \left(I_0^*(\varepsilon) + I_1^*(\varepsilon) + I_2^*(\varepsilon) \right), \tag{7.6}$$

with

$$I_0^*(\varepsilon) := 2 \int_\Omega f^*(\beta_\varepsilon^{-1} D\Psi_\varepsilon \sigma_\varepsilon)(a_2(DV) + \varepsilon m_\varepsilon) \, dx,$$

$$I_1^*(\varepsilon) := \frac{2}{\varepsilon} \int_\Omega [f^*(\beta_\varepsilon^{-1} D\Psi_\varepsilon \sigma_\varepsilon) - f^*(\overline{\sigma})] \operatorname{div} V \, dx,$$

$$I_2^*(\varepsilon) := \frac{2}{\varepsilon^2} \int_\Omega [f^*(\beta_\varepsilon^{-1} D\Psi_\varepsilon \sigma_\varepsilon) - f^*(\overline{\sigma}) - \varepsilon \langle \nabla f^*(\overline{\sigma}), (DV - \operatorname{div} V I)\overline{\sigma} \rangle - \varepsilon \lambda \operatorname{div} V \overline{u}] \, dx.$$

Now we argue in a similar way as in the strictly convex case: we apply Lemma 8.1 to the integral functional I_{f^*} and, in order to deal with $I_2^*(\varepsilon)$, we exploit the equality

$$\int_\Omega \lambda \operatorname{div} V \overline{u} \, dx = - \int_\Omega \overline{u} \operatorname{div} \eta_1 \, dx = \int_\Omega \langle \nabla f^*(\overline{\sigma}), \eta_1 \rangle \, dx;$$

thus we obtain

$$\lim_{\varepsilon \to 0} I_0^*(\varepsilon) = 2 \int_\Omega f^*(\overline{\sigma}) a_2(DV) \, dx,$$

$$\lim_{\varepsilon \to 0} I_1^*(\varepsilon) = 2 \int_\Omega \langle \nabla f^*(\overline{\sigma}), (DV - \operatorname{div} V I)\overline{\sigma} + \eta_1 \rangle \operatorname{div} V \, dx,$$

$$\lim_{\varepsilon \to 0} I_2^*(\varepsilon) = 2 \int_\Omega Q_{f^*}((DV - \operatorname{div} V I)\overline{\sigma} + \eta_1) +$$

$$+ \langle \nabla f^*(\overline{\sigma}), ((\operatorname{div} V)^2 - \operatorname{div} V DV - a_2(DV))\overline{\sigma} + (DV - \operatorname{div} V I)\eta_1 + \eta_2 \rangle \, dx.$$

In view of (7.6), by adding up the three terms above and exploiting the equality $\int_\Omega \langle \nabla f^*(\overline{\sigma}), \eta_2 \rangle \, dx = \int_\Omega \lambda \overline{u} a_2(DV) \, dx$, we infer that $\limsup_{\varepsilon \to 0} r_\varepsilon(V) \le E^*(\eta_1, V)$.

Now, since η_1 is an arbitrary vector field in $L^\infty(\Omega; \mathbb{R}^n)$ with $\operatorname{div} \eta_1 = -\lambda$ and $\eta_1 \cdot n = 0$ on Γ_N, in order to conclude the proof of the upper bound inequality (4.8) it is enough to show that every field $\eta \in X(\Omega; \mathbb{R}^n)$ satisfying the two conditions $\operatorname{div} \eta = -\lambda$ and $\eta \cdot n = 0$ on Γ_N can be approximated strongly in L^2 by a sequence of fields η^h belonging to $L^\infty(\Omega; \mathbb{R}^n)$ and satisfying the same two

conditions. To that aim we begin by noticing that, since by assumption $\partial\Omega$ is piecewise C^1, there exists a sequence of smooth fields $\tilde{\eta}^h$, with $\tilde{\eta}^h \cdot n = 0$ on Γ_N, such that $\tilde{\eta}^h \to \eta$ in $L^2(\Omega; \mathbb{R}^n)$ and $\operatorname{div} \tilde{\eta}^h \to -\lambda$ in $L^2(\Omega)$. Then, we let ϕ^h be the unique solution to the Dirichlet-Neumann problem

$$-\Delta\phi^h = \rho^h := \lambda + \operatorname{div}\tilde{\eta}^h \quad \text{in } \Omega, \qquad \phi^h = 0 \quad \text{on } \Gamma_D, \qquad \partial_n\phi^h = 0 \quad \text{on } \Gamma_N.$$

Finally we define $\eta^h := \tilde{\eta}^h + \nabla\phi^h$, and we claim that such sequence η^h has the required properties. Namely, since ρ^h is a sequence of smooth functions converging to 0 in $L^2(\Omega)$, the sequence η^h lies in $L^\infty(\Omega; \mathbb{R}^n)$, and converges to η in $L^2(\Omega; \mathbb{R}^n)$; moreover it holds $\operatorname{div}\eta^h = -\lambda$ and $\eta^h \cdot n = 0$ on Γ_N. Eventually, once now (4.8) is established, the remaining parts of the proofs given in Sections 4, 5 and 6 can be repeated as done for g strictly convex.

7.2 The p-torsion problem

Here we prove formula (3.17) for the second order shape derivative of the p-torsional rigidity functional $J_p(\Omega)$ introduced in (3.16), for $p > 2$. We recall that the first order derivative $J_p'(\Omega, V)$ exists and is given by

$$J_p'(\Omega, V) = \int_{\partial\Omega} \frac{|\nabla\overline{u}|^p}{p'} V_n \, d\mathcal{H}^{n-1} \, ,$$

where $\overline{u}$ is the unique solution in $W_0^{1,p}(\Omega)$ to $-\Delta_p\overline{u} = \lambda$ (see for instance [8, 14]). It is well-known that $\overline{u}$ is of class $C^{1,\alpha}(\Omega)$ (see [18, 43]), and it is also in $C^2(\Omega \setminus S)$, where S is the critical set $S := \{x \in \Omega : \nabla\overline{u} = 0\}$. Let us also recall that S has vanishing Lebesgue measure (see [30]), and it is compactly contained into Ω (thanks to Hopf boundary lemma).
Let us introduce the weight function $\rho := |\nabla\overline{u}|^{p-2}$, which is continuous in Ω and strictly positive outside S. Moreover, ρ^{-1} is in $L^1(\Omega)$ (see [40]).
We denote by $W_\rho^{1,2}(\Omega)$ the Hilbert space consisting of functions $v \in W_{\text{loc}}^{1,1}(\Omega)$ such that

$$\|v\|_{W_\rho^{1,2}(\Omega)} := \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L_\rho^2(\Omega)}^2 < +\infty \, ,$$

and by $H_\rho^{1,2}(\Omega)$ the completion of $C^1(\overline{\Omega})$ with respect to the above norm.
Let $\mathcal{Q}(\overline{u}, \cdot)$ denote the quadratic functional defined on $W_\rho^{1,2}(\Omega)$ by

$$\mathcal{Q}(\overline{u}, v) := \int_\Omega \langle P(\overline{u})\nabla v, \nabla v\rangle \, dx \, , \quad \text{with } \ P(\overline{u}) := |\nabla\overline{u}|^{p-2}\left(I + (p-2)\frac{\nabla\overline{u}}{|\nabla\overline{u}|} \otimes \frac{\nabla\overline{u}}{|\nabla\overline{u}|}\right) \, . \tag{7.7}$$

Notice that, since the norm of the matrix $P(\overline{u})$ is controlled by ρ, the functional $\mathcal{Q}(\overline{u}, \cdot)$ is continuous on the Hilbert space $W_\rho^{1,2}(\Omega)$. By adapting the approach developed in Section 4, we obtain the following second order differentiability result.

Theorem 7.1. *Let $\partial\Omega$ be of class C^2. Let $p > 2$ and assume that the weight $\rho := |\nabla\overline{u}|^{p-2}$ is such that*

$$H_\rho^{1,2}(\Omega) = W_\rho^{1,2}(\Omega) \, . \tag{7.8}$$

Then, the functional $J_p(\Omega)$ is twice differentiable at Ω in any direction V, and for V normal to the boundary it holds

$$J_p''(\Omega, V) = -\frac{1}{p}\int_{\partial\Omega} V_n^2 \left(p\lambda\partial_n\overline{u} + |\partial_n\overline{u}|^p H_{\partial\Omega}\right) d\mathcal{H}^{n-1} - \inf_{\substack{v \in W_\rho^{1,2}(\Omega) \\ v = -V_n\partial_n\overline{u} \text{ on } \partial\Omega}} \mathcal{Q}(\overline{u}, v) \, . \tag{7.9}$$

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Remark 7.2. Notice that, since ρ is strictly positive in a neighborhood of $\partial\Omega$, the space $W^{1,2}_\rho(\Omega)$ is embedded into $H^1_{\mathrm{loc}}(\Omega \setminus S)$, and hence the trace operator is well defined from $W^{1,2}_\rho(\Omega)$ into $H^{1/2}(\partial\Omega)$. In particular, the infimum in (7.9) is well-defined since, by the C^2 regularity of $\overline{u}$ in $\Omega \setminus S$, the function $\langle V, \nabla\overline{u}\rangle$ belongs to $H^1_{\mathrm{loc}}(\Omega \setminus S)$. Moreover, the infimum can be equivalently taken in $H^1(\Omega)$ (as written in Example 3.9).

Remark 7.3. Let us add a few comments on assumption (7.8) and related bibliographical references. As general texts on weighted Sobolev spaces, we refer to [28, 34]. Moreover, we address to the paper [13] for a counterexample showing that the L^1-summability property of ρ^{-1} is not strong enough to ensure the validity of (7.8). A sufficient condition would be $\rho \in \mathcal{A}_2$, where $\mathcal{A}_2$ is the Muckenhoupt class of functions satisfying $\sup_B \left(\frac{1}{|B|} \int_B \rho \right)\left(\frac{1}{|B|} \int_B \rho^{-1} \right) < +\infty$, where B varies among balls in $\mathbb{R}^n$ (see for instance [26]), but such kind of regularity result seems hard to prove. On the other hand, equality (7.8) holds true if the set S has vanishing capacity in $H^{1,2}_\rho(\Omega)$, namely if there exists a sequence of smooth functions $\alpha_\varepsilon : \Omega \to [0,1]$ which are equal to 1 in a ε-neighborhood of S and satisfy $\lim_\varepsilon \int_\Omega |\nabla\alpha_\varepsilon|^2 \rho \, dx = 0$. Indeed in this case, given $w \in W^{1,2}_\rho(\Omega)$ with $w = -\overline{\theta}_V$ on $\partial\Omega$, it is possible to construct an approximating sequence $\{w_\varepsilon\} \in H^{1,2}_\rho(\Omega)$ with $w_\varepsilon = -\overline{\theta}_V$ on $\partial\Omega$: it is enough to take $w_\varepsilon := (1 - \alpha_\varepsilon)w$ (which belong to $H^1(\Omega)$, and hence to $H^{1,2}_\rho(\Omega)$). In particular, our assumption (7.8) turns out to be satisfied whenever S is a singleton. Concerning the geometry of S recall that, if Ω is convex, the function $\overline{u}$ is power concave [39], and hence S agrees with the set where $\overline{u}$ assumes its maximum. If in addition Ω is *strictly* convex, it is likely true that S is reduced to a singleton (*cf.* [5, 27]), or at least that it is of vanishing capacity, so that (7.8) holds true.

Proof of Theorem 7.1. Since shape derivatives only depend on the behavior of the deformation field on the boundary, with no loss of generality we may choose V vanishing in a neighborhood of S (recall that $S \subset\subset \Omega$). Then, in order to show the equality (7.9), we proceed along the same scheme adopted in Section 4, namely we show that the lower and upper limits as $\varepsilon \to 0$ of the differential quotients $r_\varepsilon(V)$ defined in (4.1) are bounded respectively from below and from above by the same quantity, which is precisely the r.h.s. of formula (7.9). For convenience, we divide the remaining of the proof in three steps.

Step 1 (lower bound): Thanks to the assumption $p \geq 2$, the following lower bound can be achieved by arguing exactly as done in the proof the inequality (4.7) in Lemma 4.2:

$$\liminf_{\varepsilon \to 0} r_\varepsilon(V) \geq m_\infty := - \inf_{w \in C^\infty_0(\Omega)} E(w, V)\,,$$

where $E(w, V)$ is defined according to (4.4) by taking therein $f(z) = |z|^p/p$, and $g(v) = -\lambda v$. We claim that the field $B(\overline{u}, V)$ introduced in (3.2) still satisfies Proposition 3.3, namely

$$\operatorname{div} B(\overline{u}, V) = -\lambda \operatorname{div} V \quad \text{in } \mathcal{D}'(\Omega)\,. \tag{7.10}$$

The proof of the above equality cannot be repeated as done in Section 5, because the minimizer $\overline{u}$ of the functional $J_p(\Omega)$ satisfies the regularity condition $\overline{u} \in W^{2,2}(\Omega)$ only for $p \in (1,3)$; nevertheless, for $p \geq 3$, one has that $\overline{u} \in W^{2,q}(\Omega)$ for any $q < (p-1)/(p-2)$ (see [15, Proposition 2.2]). Thus $\nabla\overline{u}$ is always in $W^{1,1}(\Omega)$, and the distributional Hessian of $\overline{u}$ appearing in the definition $B(\overline{u}, V)$ is well-defined as a function in $L^1(\Omega)$. Moreover, the function $\overline{u}$ satisfies the crucial regularity condition $|\nabla\overline{u}|^{p-2}\nabla\overline{u} \in W^{1,2}(\Omega)$ (see [41, Corollary 2.1]), which enables us to obtain the proof of (7.10) as follows. Recall that $\overline{\sigma} = \nabla f(\nabla\overline{u}) = |\nabla\overline{u}|^{p-2}\nabla\overline{u}$; then, since $\nabla f(\cdot)$ is a locally Lipschitz function and $\nabla\overline{u} \in L^\infty(\Omega) \cap W^{1,1}(\Omega)$, we can apply the chain rule and we get $D\overline{\sigma} = \nabla^2 f(\nabla\overline{u})\nabla^2\overline{u}$ a.e. in Ω.

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Thus the field $B(\overline{u}, V)$ may be rewritten in terms of $\overline{\sigma}$ as $B(\overline{u}, V) = D\overline{\sigma}V - (DV - \operatorname{div} V I)\overline{\sigma}$ and in particular it belongs to $L^2(\Omega; \mathbb{R}^n)$. It is then straightforward to obtain (7.10). Namely, for any test function $\varphi \in C_c^\infty(\Omega)$, it holds

$$
\begin{aligned}
\langle \operatorname{div}(D\overline{\sigma}V), \varphi \rangle &= -\langle D\overline{\sigma}V, \nabla\varphi \rangle = -\langle \partial_k \overline{\sigma}_i V_k, \partial_i \varphi \rangle = \langle \overline{\sigma}_i \partial_k V_k, \partial_i \varphi \rangle + \langle \overline{\sigma}_i V_k, \partial_{ik}^2 \varphi \rangle \\
&= \langle -\operatorname{div}(\overline{\sigma}\operatorname{div} V) + \operatorname{div}(V \operatorname{div}\overline{\sigma}) + \operatorname{div}(DV\overline{\sigma}), \varphi \rangle \\
&= \langle \operatorname{div}[(DV - \operatorname{div} V I)\overline{\sigma}], \varphi \rangle - \langle \lambda \operatorname{div} V, \varphi \rangle,
\end{aligned}
$$

where we have used the equality $-\operatorname{div}\overline{\sigma} = \lambda$ (holding by (2.5) applied with $g(t) = -\lambda t$).

Now, we are in a position to rewrite m_∞ similarly as in Theorem 3.4 (ii). Indeed, one can check that the proof of Theorem 3.4 (ii) given in Section 6 continues to work, thanks in particular to the identity (7.10) and to the $W^{2,q}$ regularity of $\overline{u}$ for $q < (p-1)/(p-2)$). Thus, defining $\mathcal{C}_D(\overline{u}, V)$ as in (3.10) (with $f(z) = |z|^p/p$), letting $\mathcal{Q}(\overline{u}, \cdot)$ be given by (7.7), and setting $\overline{\theta}_V = \langle V, \nabla\overline{u} \rangle$, we obtain

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} r_\varepsilon(V) \geq m_\infty &= \int_{\partial\Omega} \mathcal{C}_D(\overline{u}, V) \, d\mathcal{H}^{n-1} - \inf_{w \in C_0^\infty(\Omega)} \mathcal{Q}(\overline{u}, w - \overline{\theta}_V) \\
&= \int_{\partial\Omega} \mathcal{C}_D(\overline{u}, V) \, d\mathcal{H}^{n-1} - \inf_{w \in H_{\rho,0}^{1,2}(\Omega)} \mathcal{Q}(\overline{u}, w - \overline{\theta}_V) \, ;
\end{aligned}
\tag{7.11}
$$

here $H_{\rho,0}^{1,2}(\Omega)$ denotes the space of functions in $H_\rho^{1,2}(\Omega)$ vanishing at $\partial\Omega$, and the last equality follows from the continuity of $\mathcal{Q}(\overline{u}, \cdot)$ on $W_\rho^{1,2}$.

Step 2 (upper bound): Due to the growth of order $p' \in (1, 2]$ of the Fenchel conjugate $f^*(z^*) = |z^*|^{p'}/p'$, in order bound from above $r_\varepsilon(V)$, we need to set up an approximation argument. Roughly speaking, we cover almost all Ω with a suitable family of subdomains of Ω in which the Hessian of f^* is bounded from above, and then we pass to the limit. More precisely, as $\nabla\overline{u}$ is a continuous function which is strictly positive outside S, we can construct increasing sequence of open sets $\Omega_h \uparrow (\Omega \setminus S)$ such that, for every $h \in \mathbb{N}$, $|\nabla\overline{u}| > 1/h$ in Ω_h, and the boundary $\Gamma_h := \partial\Omega_h \setminus \partial\Omega$ is smooth and has a positive distance from $\partial\Omega$. For every fixed h, we follow the same procedure adopted for the proof of the inequality (4.8) in Lemma 4.2, in the variant when g is linear described in §7.1. We choose the fields η_k appearing in (7.3) as done in (7.4), with the additional condition that $\eta_k = 0$ in $\Omega \setminus \Omega_k$ (note that this is possible thanks to the assumption made on the support of the deformation field V). We infer that, for every fixed h,

$$
\limsup_\varepsilon r_\varepsilon(V) \leq m_h := \inf_{\substack{\eta \in X(\Omega; \mathbb{R}^n), \, \operatorname{div}\eta = -\lambda \operatorname{div} V \text{ in } \Omega \\ \eta = 0 \text{ in } \Omega \setminus \Omega_h}} E^*(\eta, V),
\tag{7.12}
$$

where $E^*(\eta, V)$ is defined according to (7.1) (with $f^*(z^*) = |z^*|^{p'}/p'$).

By applying Lemma 8.5 and exploiting the regularity of $\overline{u}$, we may rewrite m_h in primal form as

$$
m_h = \int_{\partial\Omega} \mathcal{C}_D(\overline{u}, V) \, d\mathcal{H}^{n-1} - \inf_{w \in V_h} \mathcal{Q}_h(\overline{u}, w - \overline{\theta}_V),
\tag{7.13}
$$

where V_h denotes the subspace of $H^1(\Omega_h)$ of functions having zero trace on $\partial\Omega$, and

$$
\mathcal{Q}_h(\overline{u}, v) := 2 \int_{\Omega_h} \langle P(\overline{u})\nabla v, \nabla v \rangle \, dx.
$$

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Now, in order to bound from above the upper limit of m_h as $h \to +\infty$, we are going to bound from below the infimum appearing at the r.h.s. of (7.13). We start by noticing that such infimum can be equivalently taken in the space $W_{\rho,0}^{1,2}(\Omega)$ of functions in $W_\rho^{1,2}(\Omega)$ vanishing at $\partial\Omega$. Let $\{w_h\} \subset W_{\rho,0}^{1,2}(\Omega)$ be a sequence of minimizers for $\mathcal{Q}_h(\overline{u}, \cdot - \overline{\theta}_V)$. For every fixed $k \in \mathbb{N}$, such sequence is bounded in $H^1(\Omega_k)$, so that it admits a subsequence weakly converging in $H^1(\Omega_k)$ to some element $w^{(k)}$. By a diagonalization argument, we can choose the same subsequence for every k, and hence the restriction of $w^{(k)}$ to Ω_l for $l < k$ agrees with $w^{(l)}$. Hence we obtain a (not relabeled) subsequence of w_h and an element $\hat{w} \in W_{\rho,0}^{1,2}(\Omega)$ such that w_h converges weakly to $\hat{w}$ in $H^1(\Omega_k)$ for every k. Then, by Fatou's Lemma and monotone convergence, we get

$$\liminf_h \mathcal{Q}_h(\overline{u}, w_h - \overline{\theta}_V) \geq \mathcal{Q}(\overline{u}, \hat{w} - \overline{\theta}_V) \geq \inf_{w \in W_{\rho,0}^{1,2}(\Omega)} \mathcal{Q}(\overline{u}, w - \overline{\theta}_V) . \qquad (7.14)$$

By combining (7.12), (7.13), and (7.14), we obtain

$$\limsup_\varepsilon r_\varepsilon(V) \leq m_\infty^* := \int_{\partial\Omega} \mathcal{C}_D(\overline{u}, V) \, d\mathcal{H}^{n-1} - \inf_{w \in W_{\rho,0}^{1,2}(\Omega)} \mathcal{Q}(\overline{u}, w - \overline{\theta}_V). \qquad (7.15)$$

Step 3 (conclusion): We finally need to show that the lower and upper bounds m_∞ and m_∞^* in (7.11) and (7.15) agree, and that they are equal to the expression at the r.h.s. of (7.9). We firstly observe that, by arguing as in Remark 3.6, the integral over $\partial\Omega$ of $\mathcal{C}_D(\overline{u}, V)$ appearing in the expression (7.11) of m_∞ can be rewritten as done in (7.9) . Then it only remains to prove that no Lavrenteev phenomenon occurs for the infimum problems appearing in (7.11) and (7.15). In other words, after a translation, we are reduced to show that

$$\inf_{w \in W_\rho^{1,2}(\Omega)} \left\{ \mathcal{Q}(\overline{u}, \cdot) \,:\, w = -\overline{\theta}_V \text{ on } \partial\Omega \right\} = \inf_{w \in H_\rho^{1,2}(\Omega)} \left\{ \mathcal{Q}(\overline{u}, \cdot) \,:\, w = -\overline{\theta}_V \text{ on } \partial\Omega \right\} . \qquad (7.16)$$

The validity of (7.16) is an immediate consequence of our assumption (7.8) and of the continuity of $\mathcal{Q}(\overline{u}, \cdot)$ in $W_\rho^{1,2}(\Omega)$.

$\square$

7.3 Perspectives

A natural perspective is the possible extension of our results to the case of convex *non-smooth* integrands f and g. The main difficulty is that in this case one can no longer exploit the second order differentiability property for integral functionals stated in Lemma 8.1. However we expect that in some cases the second order shape derivative still exists, and admits a representation formula similar to (3.6), with the quadratic form $\mathcal{Q}(\overline{u}, \cdot)$ (formerly associated with the second order differentials of f and g) replaced by a suitable convex, positively 2-homogeneous, non-quadratic function. This kind of generalization would be useful in applications: for instance, the shape functional obtained by taking the function $f(z)$ equal to $\frac{|z|^2}{2} + \frac{1}{2}$ if $|z| \geq 1$ and $|z|$ if $|z| < 1$, and the function $g(v)$ equal to $-\lambda v$ (for some $\lambda \in \mathbb{R}$), is related to the optimization of thin rods in torsion regime (see [2, 9]), and studying its second order shape derivative might be helpful in order to investigate the occurrence of homogenization regions in an optimal design. (For the computation of the first order shape derivative in this case, see Example 3.9 (ii) in [8].)

Another perspective is trying to understand whether the results obtained in §7.2 may entail some useful information on the second order shape derivative of the shape functional $J_\infty(\Omega)$ obtained as

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the limit as $p \to +\infty$ of the p-torsional rigidity functionals $J_p(\Omega)$ defined in (3.16). Actually, up to constant multiple, it is well known that here holds $J_\infty(\Omega) = \int_\Omega d_{\partial\Omega}(x)\,dx$ and $\lim_p u_p(x) = d_{\partial\Omega}(x)$, where $d_{\partial\Omega}$ is the distance function from $\partial\Omega$, and u_p is the unique solution to $J_p(\Omega)$ (see [4, 7, 37]).

8 Appendix

Lemma 8.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a strongly convex function of class C^2 satisfying growth conditions of order 2 from above and below, and let I_ϕ be the integral functional defined on $L^2(\Omega; \mathbb{R}^n)$ by*

$$I_\phi(z) := \int_\Omega \phi(z(x))\,dx\,. \tag{8.1}$$

Let z_0 be a fixed vector field in $L^\infty(\Omega; \mathbb{R}^n)$ and, for $h \in L^2(\Omega; \mathbb{R}^n)$, set $q_\phi(h) := \frac{1}{2}\int_\Omega \langle \nabla^2 \phi(z_0)h, h\rangle\,dx$, and

$$\Delta_{\varepsilon,\phi}(h) := \frac{I_\phi(z_0 + \varepsilon h) - I_\phi(z_0) - \varepsilon \langle \nabla\phi(z_0), h\rangle_{L^2}}{\varepsilon^2}\,. \tag{8.2}$$

Then:

(i) *I_ϕ is of class C^1 on the Hilbert space $L^2(\Omega; \mathbb{R}^n)$, and $\nabla I_\phi(z) = \nabla\phi(z)$;*

(ii) *$\Delta_{\varepsilon,\phi}$ Mosco converges in $L^2(\Omega; \mathbb{R}^n)$ to q_ϕ as $\varepsilon \to 0$;*

(iii) *I_ϕ is second order differentiable at z_0 with respect to L^∞ variations; namely, for every $h \in L^\infty(\Omega; \mathbb{R}^n)$, it holds $\Delta_{\varepsilon,\phi}(h) \to q_\phi(h)$ as $\varepsilon \to 0$.*

Proof. For statements (i) and (ii), see [35, Proposition 4.1 and Corollary 3.2]. Let us now prove (iii). For a fixed $h \in L^\infty(\Omega; \mathbb{R}^n)$, the Mosco convergence property stated in (ii) implies that

$$q_\phi(h) \leq \liminf_{\varepsilon \to 0} \Delta_{\varepsilon,\phi}(h)\,.$$

In order to prove that $\limsup_\varepsilon \Delta_{\varepsilon,\phi}(h) \leq q_\phi(h)$, we exploit the second property given by the Mosco convergence: there exists a sequence $h_\varepsilon \in L^2(\Omega; \mathbb{R}^n)$ such that $h_\varepsilon \xrightarrow{L^2} h$ and $q_\phi(h_\varepsilon) = \lim_{\varepsilon \to 0} \Delta_{\varepsilon,\phi}(h_\varepsilon)$. We remark that the function $\Delta_{\varepsilon,\phi}$ is convex, therefore, $\Delta_{\varepsilon,\phi}(h) - \Delta_{\varepsilon,\phi}(h_\varepsilon) \leq \langle \nabla\Delta_{\varepsilon,\phi}(h), h - h_\varepsilon\rangle_{L^2}$. If we prove that, for $\varepsilon > 0$ small enough,

$$\|\nabla\Delta_{\varepsilon,\phi}(h)\|_{L^2} \leq C\,, \tag{8.3}$$

we are done, since we infer

$$\limsup_\varepsilon \Delta_{\varepsilon,\phi}(h) - q_\phi(h) = \limsup_\varepsilon \left(\Delta_{\varepsilon,\phi}(h) - \Delta_{\varepsilon,\phi}(h_\varepsilon)\right) \leq \lim_{\varepsilon \to 0} C\|h_\varepsilon - h\|_{L^2} = 0\,.$$

We conclude by proving the claim (8.3): we remark that $\nabla\Delta_\varepsilon(h) = [\nabla\phi(z_0 + \varepsilon h) - \nabla\phi(z_0)]/\varepsilon$, and hence (8.3) follows from the local boundedness of the Hessian matrix $\nabla^2\phi$ and recalling that by assumption there exists $R > 0$ such that $\|z_0 + th\|_{L^\infty} \leq R$ for t small enough. $\qquad\square$

Lemma 8.2. *Let X be a normed vector space and let X^* be its topological dual. We denote by $\langle \cdot, \cdot\rangle$ the duality product. Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then, for every $a \in X$ and $b \in X^*$, there holds*

$$\left(h(\cdot - a)\right)^*(z^*) = h^*(z^*) + \langle a, z^*\rangle\,, \qquad \left(h(\cdot) + \langle b, \cdot\rangle\right)^*(z^*) = h^*(z^* - b)\,.$$

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Proof. The statement follows straightforward by using the definition of Fenchel conjugate. $\qquad\square$

Lemma 8.3. *Let A be a positive definite $n \times n$ matrix and let Q_A denote the associated quadratic form, defined as $Q_A(x) := \frac{1}{2}\langle Ax, x \rangle$. Then $(Q_A)^* = Q_{A^{-1}}$.*

Proof. See [38, Chapter III]. $\qquad\square$

Lemma 8.4. *Let X be a Banach space and let $h : X \to \mathbb{R}$ be a strongly convex function of class C^2. Then $\mathrm{dom}(h^*)$ has nonempty interior, h^* is C^2 on the interior of $\mathrm{dom}(h^*)$, and $x^* = \nabla h(x)$ with $x \in X$ implies $x = \nabla h^*(x^*)$ and $\nabla^2 h^*(x^*) = \left(\nabla^2 h(x)\right)^{-1}$.*

Proof. See [19, Proposition 10 in Section 2 of Chapter II]. $\qquad\square$

Lemma 8.5. *Let Y, Z be Banach spaces. Let $A : Y \to Z$ be a linear operator with dense domain $D(A)$. Let $\Phi : Y \to \mathbb{R} \cup \{+\infty\}$ be convex, and $\Psi : Z \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous. Assume there exists $u_0 \in D(A)$ such that $\Phi(u_0) < +\infty$ and Ψ is continuous at $A\,u_0$. Let Z^* denote the dual space of Z, A^* the adjoint operator of A, and Φ^*, Ψ^* the Fenchel conjugates of Φ, Ψ. Then*

$$-\inf_{u \in Y}\left\{\Psi(A\,u) + \Phi(u)\right\} = \inf_{\sigma \in Z^*}\left\{\Psi^*(\sigma) + \Phi^*(-A^*\sigma)\right\}, \tag{8.4}$$

and the infimum at the right hand side is achieved.
Furthermore, $\overline{u}$ and $\overline{\sigma}$ are optimal for the l.h.s. and the r.h.s. of (8.4) respectively, if and only if there holds $\overline{\sigma} \in \partial\Psi(A\overline{u})$ and $-A^*\overline{\sigma} \in \partial\Phi(\overline{u})$.

Proof. See [6, Proposition 14]. $\qquad\square$

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